Conjectures for the delta operator expression $\Delta'_{e_k}\Delta_{h_r}e_n$

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Joint work with Andy Wilson

March 29, 2018

Partition and Tableau

- $\lambda = \lambda_1, \ldots, \lambda_k$ is a partition of n if $\lambda_1 \geq \ldots \geq \lambda_k$ and $\sum_{i=1}^{k} \lambda_i = n$, written $\lambda \vdash n$.
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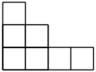
- Each partition corresponds to a Ferrers diagram. For example, $\lambda = (4, 2, 1) \vdash 7$ corresponds to  .

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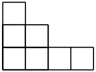
- Each partition corresponds to a Ferrers diagram. For example, $\lambda = (4,2,1) \vdash 7$ corresponds to  .

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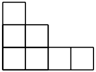
- Each partition corresponds to a Ferrers diagram. For example, $\lambda = (4,2,1) \vdash 7$ corresponds to  .

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- Column strict tableau: $\begin{array}{c|c|c|c|c|} \cline{2-2} & 5 \\ \cline{2-3} & 2 & 3 \\ \cline{2-5} \vee & 1 & 1 & 3 & 4 \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{1}{c}{\leq} \end{array}$

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- Ex.
  $f(x_1, x_2, x_3) = 3x_1x_2 + 3x_1x_3 + 3x_2x_3 + \cdots + 5x_1^2x_2 + 5x_1x_2^2 + 5x_1^2x_3 + \cdots$

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- $$s_\lambda = \sum_{\mathrm{T} \text{ a column strict tableau of shape } \lambda} X^{\mathrm{T}}.$$

Quasi-symmetric Functions

- $f(X) \in \mathbb{R}[[x]]$ is a quasi-symmetric function if for each composition $\alpha = (\alpha_1, \ldots, \alpha_k)$, the coefficient of the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ is equal to the coefficient of the monomial $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ for any strictly increasing sequence of positive integers $i_1 < i_2 < \cdots < i_k$.

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- $$F_S = \sum_{i_1 \leq i_2 \leq \ldots \leq i_n, i_j < i_{j+1} \ \ if \ \ j \in S} x_{i_1} x_{i_2} \ldots x_{i_n}$$

  is the fundamental quasi-symmetric function associated with a set $S \subset [n-1]$.

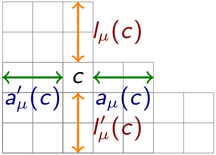
Macdonald polynomials

The Macdonald polynomial $\widetilde{H}_\mu(X; q, t)$ is a q, t-weighted symmetric function given by

$$\widetilde{H}_\mu(X; q, t) = \sum_{\sigma:\ \mu \to \mathbb{Z}_+ \text{ injective tableau}} q^{inv(\sigma)} t^{maj(\sigma)} x^\sigma.$$

The Δ operator and the ∇ operator

- Given any partition $\mu \vdash n$, we can draw the Ferrers diagram (in French notation) of μ as shown in Figure 1.



$a_\mu(c) =$ arm of c,
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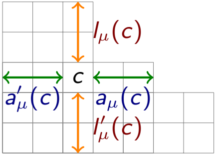

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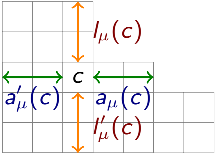

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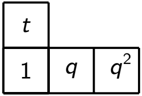

- For example, $B_{3,1} = 1 + q + q^2 + t$, $T_{3,1} = q^3 t$.

The Δ operator and the ∇ operator

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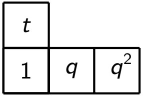

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$$\begin{aligned} \Delta_{e_2}\widetilde{H}_{3,1} &= e_2[1+q+q^2+t]\widetilde{H}_{3,1} \\ &= (q+q^2+t+q^3+qt+q^2t)\widetilde{H}_{3,1} \end{aligned}$$

- Note that $\nabla = \Delta_{e_n}$ on $\Lambda^{(n)}$.

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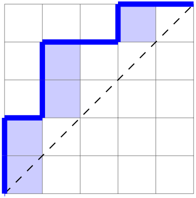
- Note that $\Delta_{e_k} = \Delta'_{e_k} + \Delta'_{e_{k-1}}$ since $e_k[X+1] = e_k[X] + e_{k-1}[X]$.
- In $\Lambda^{(n)}$, since $\Delta'_{e_n} = 0$, we have $\nabla = \Delta_{e_n} = \Delta'_{e_{n-1}}$.

Dyck Paths and Parking Functions

Definition (Dyck path)

An $n \times n$ Dyck path is a lattice path from $(0, 0)$ to (n, n) consisting of east and north steps which stays above the diagonal $y = x$.

We can get an $n \times n$ parking function by labeling the cells east of and adjacent to a north step of a Dyck path.



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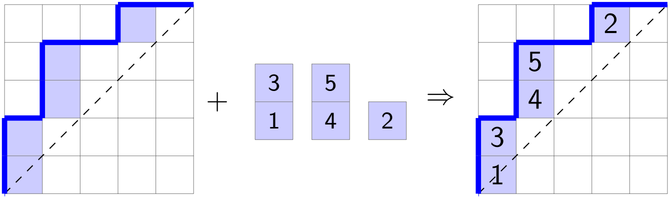


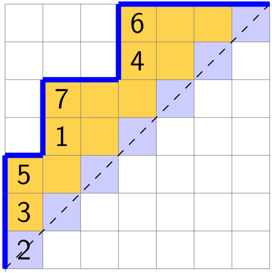
Figure 2: The construction of a parking function

Area of a Parking Function

Definition (area)

The number of full cells between the Dyck path of a parking function PF and the main diagonal is denoted by *area*(PF).

$$area(\mathrm{PF}) = \sum_{i=1}^{n} a_i(\mathrm{PF}).$$



i	$a_i(\mathrm{PF})$	$d_i(\mathrm{PF})$
7	3	0
6	2	0
5	3	1
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$\mathrm{area}(\mathrm{PF}) = 13$

Figure 3: A $(7,7)$-Parking Function

Dinv of a Parking Function

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Then,
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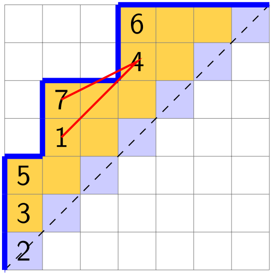

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$\mathrm{dinv}(\mathrm{PF}) = 2$

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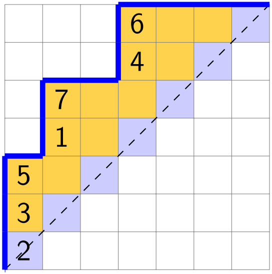
Statistics of an (n, n)-PF

- word σ: reading cars from highest → lowest diagonal.
  $\sigma(\mathrm{PF}) = 6741532$.
- $\mathrm{iDes}(\mathrm{PF}) = \mathrm{iDes}(\sigma(\mathrm{PF})) = \{i \in \sigma : i + 1 \leftarrow i\}$.
  $\mathrm{iDes}(\mathrm{PF}) = \{2, 3, 5\}$.



weight

$$= t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{dinv}(\mathrm{PF})} F_{\mathrm{iDes}(\mathrm{PF})}$$

$$= t^{13} q^2 F_{2,3,5}$$

Figure 3: A $(7, 7)$-Parking Function

The Ring of Diagonal Harmonics

Let $\mathbf{X} = x_1, x_2, \ldots, x_n$ and $\mathbf{Y} = y_1, y_2, \ldots, y_n$ be two sets of n variables. The ring of Diagonal harmonics consists of those polynomials in $\mathbb{Q}[\mathbf{X}, \mathbf{Y}]$ which satisfy the following system of differential equations

$$\partial_{x_1}^{a} \partial_{y_1}^{b} f(\mathbf{x}, \mathbf{y}) + \partial_{x_2}^{a} \partial_{y_2}^{b} f(\mathbf{x}, \mathbf{y}) + \ldots + \partial_{x_n}^{a} \partial_{y_n}^{b} f(\mathbf{x}, \mathbf{y}) = 0,$$

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for each pair of integers a and b, such that $a + b > 0$.

Haiman proved that the ring of diagonal harmonics has dimension $(n+1)^{n-1}$.

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The classical shuffle conjecture (now the Shuffle Theorem) of Haglund, Haiman, Loehr, Remmel, and Ulyanov(2005) gives a well-studied combinatorial expression for the bigraded Frobenius characteristic of the ring of diagonal harmonics:

Theorem (Carlson-Mellit)

For all $n \geq 0$,

$$\nabla e_n = \sum_{\mathrm{PF} \in \mathcal{PF}_n} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} F_{\mathrm{iDes(PF)}}.$$

The theorem was proved by Carlson and Mellit in 2015.

The Shuffle Theorem – generalizations

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The Delta Conjecture of Haglund, Remmel and Wilson is another well studied extension of the shuffle Theorem. The Delta Conjecture has two versions, rise version and valley version.

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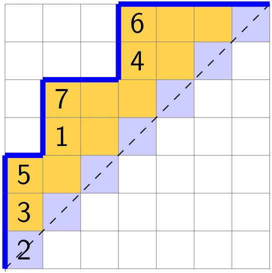

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The Delta Conjecture – Rise Version

Delta Conjecture, Rise Version (Haglund, Remmel and Wilson)

$$\Delta'_{e_k} e_n = \sum_{\mathrm{PF} \in \mathcal{PF}_n} t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{dinv}(\mathrm{PF})} F_{\mathrm{iDes}(\mathrm{PF})} \prod_{i \in \mathrm{Rise}(\mathrm{PF})} (1 + \frac{z}{t^{a_i(\mathrm{PF})}})\Big|_{z^{n-k-1}}.$$

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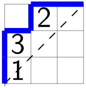
For example, the parking function 

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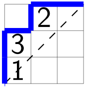
For example, the parking function 

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Taking the coefficient of z^{n-k-1} is like deleting $n-k-1$ rows from double rise to compute area.

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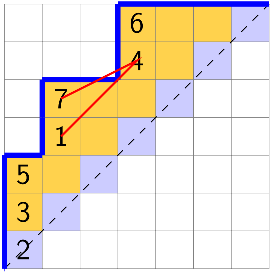

i	$a_i(\mathrm{PF})$	$d_i(\mathrm{PF})$
7	3	0
6	2	0
5	3	1
4	2	1
3	2	0
2	1	0
1	0	0

$\mathrm{dinv}(\mathrm{PF}) = 2$

Figure 3: A $(7, 7)$-Parking Function

The Delta Conjecture – Valley Version

Delta Conjecture, Valley Version (Haglund, Remmel and Wilson)

$$\Delta'_{e_k} e_n = \sum_{\mathrm{PF} \in \mathcal{PF}_n} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} F_{\mathrm{iDes(PF)}} \prod_{i \in \mathrm{Val(PF)}} (1 + \frac{z}{q^{d_i(\mathrm{PF})+1}}) \Big|_{z^{n-k-1}}.$$

Here $\mathrm{Val(PF)} = \{i | a_i < a_{i-1} \text{ or } a_i = a_{i-1} \text{ and } l_i > l_{i-1}\}$ is the collection of contractible valley of the path of the parking function PF.

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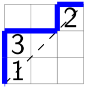
For example, the parking function



contributes $t^1 q^2 F_{1,1,1}(1 + z/q)\big|_z$ which equals $qtF_{1,1,1}$ to $\Delta'_{e_1} e_3$.

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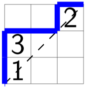
For example, the parking function  contributes

$t^1 q^2 F_{1,1,1}(1+z/q)\big|_z$ which equals $qtF_{1,1,1}$ to $\Delta'_{e_1} e_3$.

Taking the coefficient of z^{n-k-1} is like deleting $n-k-1$ rows from contractible valley and then deduct $n-k-1$ to compute dinv.

What is known for $\Delta'_{e_k} e_n$? $q = 0$ or $t = 0$ case

The Delta Conjecture is not yet proved, but a lot of cases are proved. We start with $q = 0$ or $t = 0$ case.

What is known for $\Delta'_{e_k} e_n$? $q = 0$ or $t = 0$ case

The Delta Conjecture is not yet proved, but a lot of cases are proved. We start with $q = 0$ or $t = 0$ case.

We let $\mathrm{Rise}_{n,k}(X; q, t)$ and $\mathrm{Val}_{n,k}(X; q, t)$ be the **RHS** (the combinatorial side) of the rise version and the valley version of the conjecture. Then there are many connections of the combinatorial side with ordered multiset partition statistics distributions.

Ordered Set and Multiset Partitions

- The ordered set partitions of n with k blocks are partitions of $\{1, \ldots, n\}$ into k ordered subsets, denoted $\mathcal{OP}_{n,k}$.
- For example, $27/145/36 \in \mathcal{OP}_{7,3}$.

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- Further, given a weak composition $\alpha = \{\alpha_1, \ldots, \alpha_n\}$, we let $\mathcal{OP}_{\alpha,k}$ denote the set of partitions of multiset $\{i^{\alpha_i} : 1 \leq i \leq n\}$ into k ordered sets.
- For example, $13/14/345 \in \mathcal{OP}_{\{2,0,2,2,1\},3}$.

Multiset Partition Statistics

- Given $\pi \in \mathcal{OP}_{\alpha,k}$, the inversion of π, denoted $\mathrm{inv}(\pi)$ is the number of pairs $a > b$ such that a's block is strictly to the left of b's block and b is the minimum of that block.
- For example, 15/23/4 has 2 inversions, caused by $(5,2)$ and $(5,4)$.

Multiset Partition Statistics

- Given $\pi = \pi_1 \backslash \cdots \backslash \pi_k \in \mathcal{OP}_{\alpha,k}$, let π_i^h be the h^{th} smallest number in π_i. The diagonal inversion of π, denoted $\mathrm{dinv}(\pi)$, is defined by

$$\begin{aligned} \mathrm{dinv}(\pi) &= |\{(h,i,j) : 1 \leq i < j \leq k, \pi_i^h > \pi_j^h\}| \\ &\quad +|\{(h,i,j) : 1 \leq i < j \leq k, \pi_i^h < \pi_j^{h+1}\}| \end{aligned}$$

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- For example, $\underline{\bar{2}}\ \hat{4}/\bar{1}\ \underline{\hat{3}}\ 4/2$ has 3 diagonal inversions caused by $(2,1)$, $(4,3)$ and $(2,3)$.
- Two other statistics called maj and $\mathrm{minimaj}$ are also defined on ordered multiset partitions.

Connection between PF's and OMP's

Let $D^{\mathrm{stat}}_{\alpha,k}(q) = \sum_{\pi \in \mathcal{OP}_{\alpha,k}} q^{\mathrm{stat}(\pi)}$, Haglund, Remmel and Wilson showed that

Theorem (Haglund, Remmel and Wilson)

$$\begin{aligned}
\mathrm{Rise}_{n,k}(X;q,0)|_{M_\alpha} &= D^{\mathrm{dinv}}_{\alpha,k+1}(q),\\
\mathrm{Rise}_{n,k}(X;0,q)|_{M_\alpha} &= D^{\mathrm{maj}}_{\alpha,k+1}(q),\\
\mathrm{Val}_{n,k}(X;q,0)|_{M_\alpha} &= D^{\mathrm{inv}}_{\alpha,k+1}(q),\\
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\end{aligned}$$

Thus, we can work on ordered set partition instead of on parking functions for the combinatorial side of the Delta Conjecture when q or $t = 0$.

Equivalence of Rise and Valley Version when $q = 0$ or $t = 0$

The following theorem due to the work of Haglund, Remmel, Rhoades and Wilson shows that the Rise and the Valley version of the conjecture are equivalent when q or $t = 0$.

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As a consequence,

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Delta Conjecture at $q = 0$ Solved

- Haglund, Rhoades and Shimozono were able to represent $\mathrm{Rise}_{n,k}(X; q, 0)$ (up to q-reverse and ω action) as the graded Frobenius character of ring $R_{n,k}$ which is a generalization of the coinvariant algebra. They have a nice expansion in *dual Hall-Littlewood polynomials* $Q'_\lambda(X; q)$.

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- **Catalan case** of the conjecture is proved by Zabrocki.

Our Problem: Conjectures for $\Delta'_{e_k}\Delta_{h_r}e_n$

Our main focus is the combinatorics of $\Delta'_{e_k}\Delta_{h_r}e_n$.

The problem was initially proposed by Haglund, Remmel and Wilson, who gave a rise version conjecture for $\Delta'_{e_k}\Delta_{h_r}e_n$, whose combinatorial side was decorated parking functions with blank valleys.

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The problem was initially proposed by Haglund, Remmel and Wilson, who gave a rise version conjecture for $\Delta'_{e_k}\Delta_{h_r}e_n$, whose combinatorial side was decorated parking functions with blank valleys.

We extend the conjecture into both rise version and valley version, and prove some combinatorics about the combinatorial side.

Statistics of Parking Functions with Blank Valleys

- Given a path P, valley of P is defined by

$$\mathrm{valley}(P) = \{i | a_i \leq a_{i-1} \textit{ and } i \geq 2\}.$$

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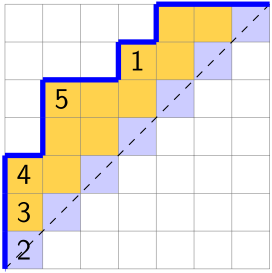
- We say that a parking function has r blank valley if there are r valleys not receiving a label.

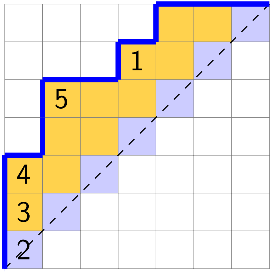


i	$a_i(\mathrm{PF})$	$d_i(\mathrm{PF})$
7	2	0
6	2	0
5	3	2
4	2	1
3	2	0
2	1	0
1	0	0

Figure 5: A $(7,7)$-Parking Function with 2 blank valleys

Statistics of Parking Functions with Blank Valleys

- The areas a_i are defined as before.
- The dinvs d_i are calculated by labeling the blank valleys with 0s.
- $\mathrm{iDes}(\mathrm{PF})$ is the iDes for the nonblank labels.



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3	2	0
2	1	0
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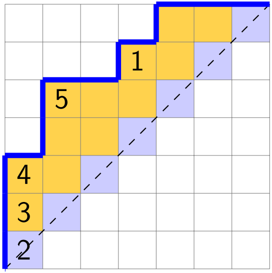
Statistics of Parking Functions with Blank Valleys

- The double rise rows are defined as before.
- The contractible valley rows are selected by labeling the blank valleys with 0s.
- Let $\mathcal{PF}^{\mathrm{Blank}}_{N,r}$ be the set of word parking functions of size N with r blank valleys.



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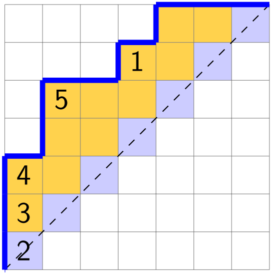
Conjecture for $\Delta'_{e_k}\Delta_{h_r}e_n$ – Rise Version

Conjecture for $\Delta'_{e_k}\Delta_{h_r}e_n$, Rise Version

For any positive integers n, k, and r with $k < n$,

$$\Delta'_{e_k}\Delta_{h_r}e_n = \sum_{\mathrm{PF}\in\mathcal{PF}^{\mathrm{Blank}}_{n+r,r}} t^{\mathrm{area}(\mathrm{PF})}q^{\mathrm{dinv}(\mathrm{PF})}F_{\mathrm{iDes}(\mathrm{PF})}\prod_{i\in\mathrm{Rise}(\mathrm{PF})}\left(1+\frac{z}{t^{a_i(\mathrm{PF})}}\right)\Bigg|_{z^{n-k-1}}.$$



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7	2	0
6	2	0
5	3	2
4	2	1
3	2	0
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1	0	0

weight$(\mathrm{PF}) = t^6q^3F_{2,3,4}$

to $\Delta'_{e_1}\Delta_{h_2}e_5$

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Conjecture for $\Delta'_{e_k}\Delta_{h_r}e_n$ – Valley Version

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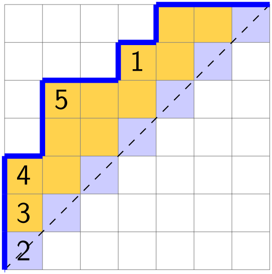

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$weight(\mathrm{PF}) = 0$

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Things that are known for $\Delta'_{e_k}\Delta_{h_r}e_n$, $q = 0$ or $t = 0$ case

We have some combinatorial progress on $q = 0$ and $t = 0$ case.

We let $\mathrm{Rise}_{n,k,r}(X; q, t)$ and $\mathrm{Val}_{n,k,r}(X; q, t)$ be the **RHS** (the combinatorial side) of the rise version and the valley version of the conjecture. Then there are many connections of the combinatorial side with ordered multiset partition (with zeros) statistics distributions.

Ordered Set and Multiset Partitions with zeros

We form sets of ordered multiset partitions.

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It is not hard to check that the four statistics, inversion, diagonal inversion, maj and minimaj are well defined on sets $\mathcal{OP}_{\alpha,k,r}$.

Connection between PF's and OMP's

Let $D^{\mathrm{stat}}_{\alpha,k,r}(q) = \sum_{\pi \in \mathcal{OP}_{\alpha,k,r}} q^{\mathrm{stat}(\pi)}$, we prove using similar techniques that

Theorem (Q. – Wilson)

$$\begin{aligned}
\mathrm{Rise}_{n,k,r}(X;q,0)|_{M_\alpha} &= D^{\mathrm{dinv}}_{\alpha,k+1,r}(q),\\
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Thus, we can work on ordered set partition instead of on parking functions for the combinatorial side of the $\Delta'_{e_k}\Delta_{h_r}e_n$ Conjecture when q or $t = 0$.

Equivalence of Rise and Valley Version when $q = 0$ or $t = 0$

We can show that the Rise and the Valley version of the conjecture are equivalent when q or $t = 0$ by similar but more complicated techniques that,

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Some Attempt on the Combinatorial Side

- If we allow 0 to appear in the last set of a multiset partition, in other word, if we allow blank valley and blank row 1, we are actually getting $h_r^{\perp}\mathrm{Rise}_{n,k}(X; q, t)$.

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- Since $h_r^\perp \mathrm{Rise}_{n,k}(X;q,t)$ should be symmetric and Schur positive as long as the Delta Conjecture is true, we can think about the complement

  – we fix a 0 in the last part of an ordered multiset partition,
  or we fix a blank label in the first row.

Open Problems

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- Find Q'_λ-expansion of $\omega\Delta'_{e_k}\Delta_{h_r}e_n|_{q=0}$.
- Can we find **a module** such that the graded Frobenius Character is equal to $\mathrm{Rise}_{n,k,r}(X;q,0)$?

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- Are there Square Path Conjecture analogue of the Delta Conjecture?

Thank You!