Conjectures for the delta operator expression $\Delta'_{e_k}\Delta_{h_r}e_n$

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The Ring of Diagonal Harmonics

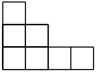
Let $\mathbf{X} = x_1, x_2, \ldots, x_n$ and $\mathbf{Y} = y_1, y_2, \ldots, y_n$ be two sets of n variables. The ring of Diagonal harmonics consists of those polynomials in $\mathbb{Q}[\mathbf{X}, \mathbf{Y}]$ which satisfy the following system of differential equations

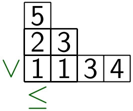
$$\partial_{x_1}^a \partial_{y_1}^b f(\mathbf{x}, \mathbf{y}) + \partial_{x_2}^a \partial_{y_2}^b f(\mathbf{x}, \mathbf{y}) + \ldots + \partial_{x_n}^a \partial_{y_n}^b f(\mathbf{x}, \mathbf{y}) = 0,$$

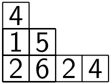
for each pair of integers a and b, such that $a + b > 0$.

Haiman proved that the ring of diagonal harmonics has dimension $(n+1)^{n-1}$.

Partition and Tableau

- $\lambda = \lambda_1, \ldots, \lambda_k$ is a partition of n if $\lambda_1 \geq \ldots \geq \lambda_k$ and $\sum_{i=1}^{k} \lambda_i = n$, written $\lambda \vdash n$.
- Ex. $\lambda \vdash 3 : (3), (2,1), (1,1,1)$.
- Each partition corresponds to a Ferrers diagram. For example, $\lambda = (4,2,1) \vdash 7$ corresponds to  .

  We can fill the cells of the Ferrers diagram with integers.
- Column strict tableau: 

- Injective tableau: $\lambda \to \mathbb{Z}_+$, 

Symmetric Functions

- $\mathcal{S}_n = \{\sigma : \sigma$ is a permutation of $[n]\}$ is the n^{th} symmetric group.
- $f(X) \in \mathbb{R}[[x]]$ is a symmetric function if $f(X) = f(\sigma(X))$ for any permutation σ.
- Ex. $f(x_1, x_2, x_3) = 3x_1x_2 + 3x_1x_3 + 3x_2x_3 + \cdots + 5x_1^2x_2 + 5x_1x_2^2 + 5x_1^2x_3 + \cdots$
- The ring of symmetric functions has several bases: $\{s_\lambda\}, \{e_\lambda\}, \ldots$.
- $e_n = \sum_{i_1 < \cdots < i_n} x_{i_1}x_{i_2}\cdots x_{i_n}$, and $e_\lambda = e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_k}$.
- $$s_\lambda = \sum_{\mathrm{T} \text{ a column strict tableau of shape } \lambda} X^{\mathrm{T}}.$$

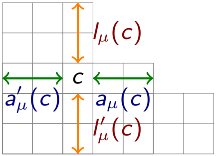
Quasi-symmetric Functions

- $f(X) \in \mathbb{R}[[x]]$ is a quasi-symmetric function if for each composition $o(\alpha_1, \ldots, \alpha_k)$, the coefficient of the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ is equal to the coefficient of the monomial $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ for any strictly increasing sequence of positive integers $i_1 < i_2 < \cdots < i_k$.
- $$F_S = \sum_{i_1 \leq i_2 \leq \ldots \leq i_n, i_j < i_{j+1} \ \ if \ \ j \in S} x_{i_1} x_{i_2} \ldots x_{i_n}$$

  is the fundamental quasi-symmetric function associated with a set $S \subset [n-1]$.

Arm and Leg of a Cell

Given any partition $\mu \vdash n$, we can draw the Ferrers diagram (in French notation) of μ as shown in Figure 1.



$$\begin{aligned} a_\mu(c) &= \text{arm of } c, \\ a'_\mu(c) &= \text{coarm of } c, \\ l_\mu(c) &= \text{leg of } c, \\ l'_\mu(c) &= \text{coleg of } c. \end{aligned}$$

Figure 1: The Young tableau of the partition $(7, 7, 5, 3, 3)$

Then for each cell $c \in \mu$, we have the arm $a_\mu(c)$, the coarm $a'_\mu(c)$, the leg $l_\mu(c)$, and the coleg $l'_\mu(c)$ of c.

Macdonald polynomials

- The Macdonald polynomial $\widetilde{H}_\mu(X;q,t)$ is a q,t-weighted symmetric function given by

$$\widetilde{H}_\mu(X;q,t) = \sum_{\sigma\colon\ \mu\to\mathbb{Z}_+ \text{ injective tableau}} q^{inv(\sigma)} t^{maj(\sigma)} x^\sigma.$$

- The symmetric function operator nabla ∇ is the eigenoperator on Macdonald polynomials defined by Bergeron and Garsia where

$$\nabla \widetilde{H}_\mu(X;q,t) = T_\mu \widetilde{H}_\mu(X;q,t).$$

Here $T_\mu = \prod_{c\in\mu} q^{a'_\mu(c)} t^{l'_\mu(c)}$.

Dyck Paths and Parking Functions

Definition (Dyck path)

An $n \times n$ Dyck path is a lattice path from $(0,0)$ to (n,n) consisting of east and north steps which stays above the diagonal $y = x$.

We can get an $n \times n$ parking function by labeling the cells east of and adjacent to a north step of a Dyck path.

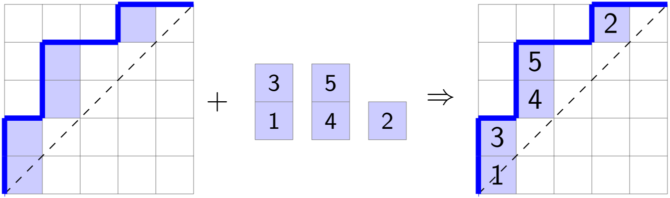


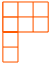
Figure 2: The construction of a parking function

Area of a Dyck Path

Definition (area)

The number of full cells between an (n, n)-Dyck path Π and the main diagonal is denoted $\mathit{area}(\Pi)$.

The collection of cells above a Dyck path Π forms an the Ferrers diagram (English) of a partition $\lambda(\Pi)$.

Ex. $\lambda(\Pi) = (3, 3, 1, 1)$, .

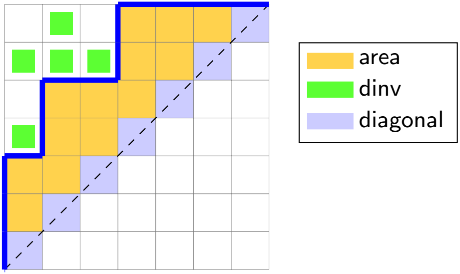


Figure 3: A $(7, 7)$-Dyck path

Dinv of a Dyck Path

Definition (dinv)

The dinv of an (n, n)-Dyck path Π is given by

$$\mathrm{dinv}(\Pi) = \sum_{c \in \lambda(\Pi)} \chi\left(\frac{arm(c)}{leg(c)+1} \leq 1 < \frac{arm(c)+1}{leg(c)}\right).$$

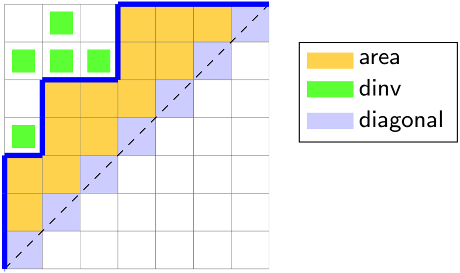
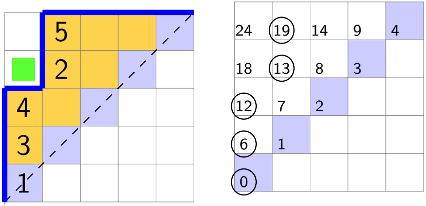


Figure 3: A $(7, 7)$-Dyck path

Statistics of an (n, n)-PF

- $\mathrm{area}(\mathrm{PF}) = \mathrm{area}(\Pi(\mathrm{PF})) = 8$,
- rank of a cell is $\mathrm{rank}(x, y) = (n+1)y - nx$,
- $\mathrm{dinv}(\mathrm{PF}) = \sum_{cars\ i<j} \chi(rank(i) < rank(j) \leq rank(i)+n) = 0$,
- word σ: reading cars from highest $\rightarrow$ lowest rank.
  $\sigma(\mathrm{PF}) = 52431$.
- $\mathrm{ides}(\sigma) = \{i \in \sigma : i+1 \leftarrow i\}$, $\mathrm{pides}(\sigma)$ is the composition corresponding to $\mathrm{ides}(\sigma)$. $\mathrm{ides}(\mathrm{PF}) = \{1, 3, 4\}$ and $\mathrm{pides}(\mathrm{PF}) = \{1, 2, 1, 1\}$.



weight
$= t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{dinv}(\mathrm{PF})} F_{\mathrm{ides}(\mathrm{PF})}$
$= t^8 q^0 F_{1,3,4}$

Figure 4: A $(5, 5)$-Parking Function

Classical Shuffle Conjecture

The bigraded Frobenius characteristic of the $\mathcal{S}_n$-module (under the diagonal action) of the ring of diagonal harmonics is given by ∇e_n.

The classical shuffle conjecture of Haglund, Haiman, Loehr, Remmel, and Ulyanov(2005) gives a well-studied combinatorial expression for the bigraded Frobenius characteristic of the ring of diagonal harmonics:

Conjecture (Haglund-Haiman-Loehr-Remmel-Ulyanov)

For all $n \geq 0$*,*

$$\nabla e_n = \sum_{\mathrm{PF} \in \mathcal{PF}_n} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} F_{\mathrm{ides(PF)}}.$$

Symmetric Function Side Extension — ????

Thank You!