Special Cases in the Combinatorics of Rational Shuffle Conjecture

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The Ring of Diagonal Harmonics

Let $\mathbf{X} = x_1, x_2, \ldots, x_n$ and $\mathbf{Y} = y_1, y_2, \ldots, y_n$ be two sets of n variables. The ring of Diagonal harmonics consists of those polynomials in $\mathbb{Q}[\mathbf{X}, \mathbf{Y}]$ which satisfy the following system of differential equations

$$\partial_{x_1}^{a} \partial_{y_1}^{b} f(\mathbf{x}, \mathbf{y}) + \partial_{x_2}^{a} \partial_{y_2}^{b} f(\mathbf{x}, \mathbf{y}) + \ldots + \partial_{x_n}^{a} \partial_{y_n}^{b} f(\mathbf{x}, \mathbf{y}) = 0,$$

for each pair of integers a and b, such that $a + b > 0$.

Haiman proved that the ring of diagonal harmonics has dimension $(n+1)^{n-1}$.

Partition and Tableau

- $\lambda = \lambda_1, \ldots, \lambda_k$ is a partition of n if $\lambda_1 \geq \ldots \geq \lambda_k$ and $\sum_{i=1}^{k} \lambda_i = n$, written $\lambda \vdash n$.
- Ex. $\lambda \vdash 3 : (3), (2,1), (1,1,1)$.

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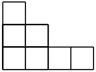
- Ex. $\lambda \vdash 3 : (3), (2, 1), (1, 1, 1)$.
- Each partition corresponds to a Ferrers diagram. For example, $\lambda = (4, 2, 1) \vdash 7$ corresponds to  .

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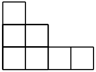
- Each partition corresponds to a Ferrers diagram. For example, $\lambda = (4,2,1) \vdash 7$ corresponds to  .

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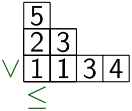

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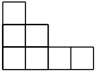
- Each partition corresponds to a Ferrers diagram. For example, $\lambda = (4,2,1) \vdash 7$ corresponds to  .

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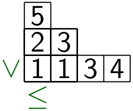
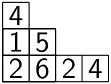
- Column strict tableau: 

- Injective tableau: $\lambda \rightarrow \mathbb{Z}_+$, 

Symmetric Functions

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- Ex. $f(x_1, x_2, x_3) = 3x_1x_2 + 3x_1x_3 + 3x_2x_3 + \cdots + 5x_1^2x_2 + 5x_1x_2^2 + 5x_1^2x_3 + \cdots$

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$$s_\lambda = \sum_{\mathrm{T} \text{ a column strict tableau of shape } \lambda} X^{\mathrm{T}}.$$

Quasi-symmetric Functions

- $f(X) \in \mathbb{R}[[x]]$ is a quasi-symmetric function if for each composition $\mathrm{o}(\alpha_1, \ldots, \alpha_k)$, the coefficient of the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ is equal to the coefficient of the monomial $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ for any strictly increasing sequence of positive integers $i_1 < i_2 < \cdots < i_k$.

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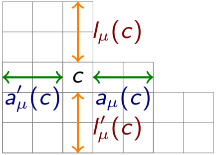
- $$F_S = \sum_{i_1 \leq i_2 \leq \ldots \leq i_n, i_j < i_{j+1} \ \ if \ \ j \in S} x_{i_1} x_{i_2} \ldots x_{i_n}$$

  is the fundamental quasi-symmetric function associated with a set $S \subset [n-1]$.

Arm and Leg of a Cell

Given any partition $\mu \vdash n$, we can draw the Ferrers diagram (in French notation) of μ as shown in Figure 1.



$$\begin{aligned} a_\mu(c) &= \text{arm of } c, \\ a'_\mu(c) &= \text{coarm of } c, \\ l_\mu(c) &= \text{leg of } c, \\ l'_\mu(c) &= \text{coleg of } c. \end{aligned}$$

Figure 1: The Young tableau of the partition $(7, 7, 5, 3, 3)$

Then for each cell $c \in \mu$, we have the arm $a_\mu(c)$, the coarm $a'_\mu(c)$, the leg $l_\mu(c)$, and the coleg $l'_\mu(c)$ of c.

Macdonald polynomials

- The Macdonald polynomial $\widetilde{H}_\mu(X;q,t)$ is a q,t-weighted symmetric function given by

$$\widetilde{H}_\mu(X;q,t) = \sum_{\sigma\colon\ \mu\to\mathbb{Z}_+ \text{ injective tableau}} q^{inv(\sigma)} t^{maj(\sigma)} x^\sigma .$$

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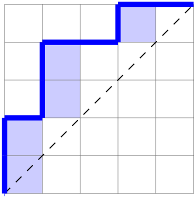
- The symmetric function operator nabla ∇ is the eigenoperator on Macdonald polynomials defined by Bergeron and Garsia where
$$\nabla \widetilde{H}_\mu(X;q,t) = T_\mu \widetilde{H}_\mu(X;q,t).$$
Here $T_\mu = \prod_{c\in\mu} q^{a'_\mu(c)} t^{l'_\mu(c)}$.

Dyck Paths and Parking Functions

Definition (Dyck path)

An $n \times n$ Dyck path is a lattice path from $(0, 0)$ to (n, n) consisting of east and north steps which stays above the diagonal $y = x$.

We can get an $n \times n$ parking function by labeling the cells east of and adjacent to a north step of a Dyck path.



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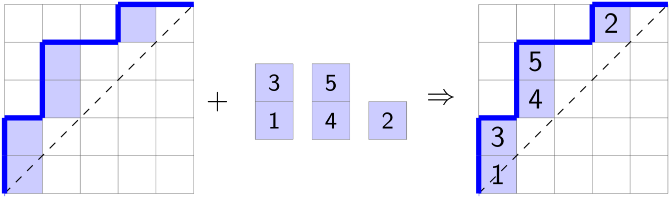


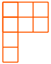
Figure 2: The construction of a parking function

Area of a Dyck Path

Definition (area)

The number of full cells between an (n, n)-Dyck path Π and the main diagonal is denoted *area*(Π).

The collection of cells above a Dyck path Π forms an the Ferrers diagram (English) of a partition $\lambda(\Pi)$.

Ex. $\lambda(\Pi) = (3, 3, 1, 1)$, .

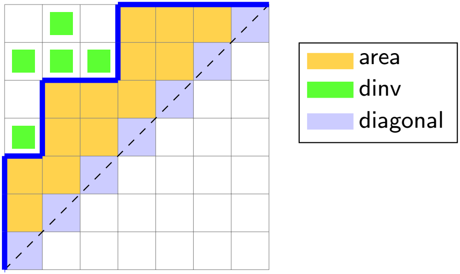


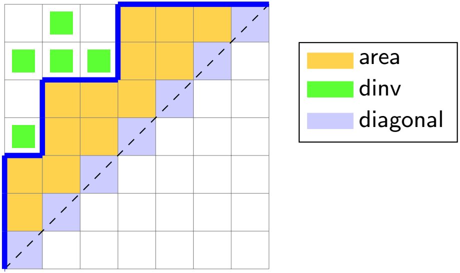
Figure 3: A $(7, 7)$-Dyck path

Dinv of a Dyck Path

Definition (dinv)

The dinv of an (n, n)-Dyck path Π is given by

$$\mathrm{dinv}(\Pi) = \sum_{c \in \lambda(\Pi)} \chi\left(\frac{arm(c)}{leg(c)+1} \leq 1 < \frac{arm(c)+1}{leg(c)}\right).$$



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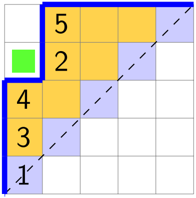
Statistics of an (n, n)-PF

- $\mathrm{area}(\mathrm{PF}) = \mathrm{area}(\Pi(\mathrm{PF})) = 8$,
- rank of a cell is $\mathrm{rank}(x, y) = (n+1)y - nx$,
- $\mathrm{dinv}(\mathrm{PF}) = \sum_{cars\ i<j} \chi(rank(i) < rank(j) \leq rank(i)+n) = 0$,
- word σ: reading cars from highest → lowest rank.
  $\sigma(\mathrm{PF}) = 52431$.
- $\mathrm{ides}(\sigma) = \{i \in \sigma : i+1 \leftarrow i\}$, $\mathrm{pides}(\sigma)$ is the composition corresponding to $\mathrm{ides}(\sigma)$. $\mathrm{ides}(\mathrm{PF}) = \{1, 3, 4\}$ and $\mathrm{pides}(\mathrm{PF}) = \{1, 2, 1, 1\}$.



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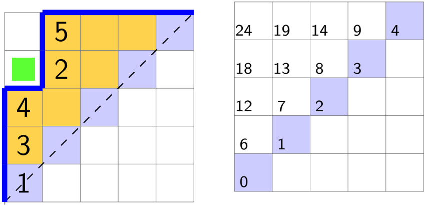

24	19	14	9	4
18	13	8	3	
12	7	2		
6	1			
0				

Figure 4: A (5, 5)-Parking Function

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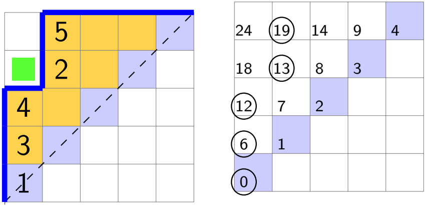

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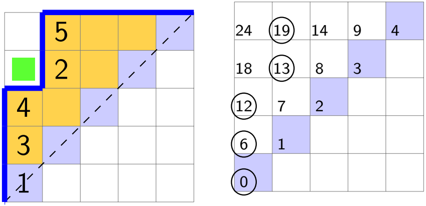

24	⑲	14	9	4
18	⑬	8	3	
⑫	7	2		
⑥	1			
⓪				

weight

$= t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{dinv}(\mathrm{PF})} F_{\mathrm{ides}(\mathrm{PF})}$

$= t^8 q^0 F_{1,3,4}$

Figure 4: A $(5, 5)$-Parking Function

Classical Shuffle Conjecture

The bigraded Frobenius characteristic of the $\mathcal{S}_n$-module (under the diagonal action) of the ring of diagonal harmonics is given by ∇e_n.

The classical shuffle conjecture of Haglund, Haiman, Loehr, Remmel, and Ulyanov(2005) gives a well-studied combinatorial expression for the bigraded Frobenius characteristic of the ring of diagonal harmonics:

Conjecture (Haglund-Haiman-Loehr-Remmel-Ulyanov)

For all $n \geq 0$,

$$\nabla e_n = \sum_{\mathrm{PF} \in \mathcal{PF}_n} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} F_{\mathrm{ides(PF)}}.$$

Symmetric Function Side Extension — $\mathrm{Q}_{m,n}$ Operators

Gorsky and Negut introduced operator $\mathrm{Q}_{m,n}$ and extended the shuffle conjecture from ∇e_n to $\mathrm{Q}_{m,n}(-1)^n$.

The main actors on the symmetric function side of the Gorsky-Negut conjecture are the operators D_k for each integer k, which were introduced in Garsia et al.(1999). The action of D_k on a symmetric function $F[X]$ is defined as

$$D_k\ F[X] = F\left[X + \frac{M}{z}\right] \sum_{i \geq 0} (-z)^i e_i[X] \Big|_{z^k},$$

where $M = (1-t)(1-q)$.

Symmetric Function Side Extension — $\mathrm{Q}_{m,n}$ Operators

We will construct a family of symmetric function operators $\mathrm{Q}_{a,b}$ for any pair of positive integers (a, b). It will be convenient to use the notation $\mathrm{Q}_{km,kn}$ with (m, n) coprime.

- For any $n \geq 0$, set $\mathrm{Q}_{1,n} = D_n$.

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- For any $n \geq 0$, set $\mathrm{Q}_{1,n} = D_n$.
- Then we will recursively define $\mathrm{Q}_{m,n}$ as follows for $m > 1$. Consider the $m \times n$ lattice with diagonal $y = \frac{n}{m}x$. Let (a, b) be the lattice point which is closest to and below the diagonal. Set $(c, d) = (m - a, n - b)$. We will write

$$\mathrm{Split}(m, n) = (a, b) + (c, d).$$

Then let

$$\mathrm{Q}_{m,n} = \frac{1}{M}[\mathrm{Q}_{c,d}, \mathrm{Q}_{a,b}] = \frac{1}{M}(\mathrm{Q}_{c,d}\mathrm{Q}_{a,b} - \mathrm{Q}_{a,b}\mathrm{Q}_{c,d}).$$

Symmetric Function Side Extension — $\mathrm{Q}_{m,n}$ Operators

Figure 5 gives an example of $\mathrm{Split}(3,5)$.

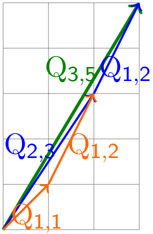

Figure 5: The geometry of $\mathrm{Split}(3,5)$

$\mathrm{Split}(3,5) = (2,3) + (1,2)$ so that $\mathrm{Q}_{3,5} = \frac{1}{M}[\mathrm{Q}_{1,2}, \mathrm{Q}_{2,3}]$.

The same procedure gives $\mathrm{Q}_{2,3} = \frac{1}{M}[\mathrm{Q}_{1,2}, \mathrm{Q}_{1,1}]$. Therefore

$$\mathrm{Q}_{3,5} = \frac{1}{M^2}[D_2, [D_2, D_1]] = \frac{1}{M^2}(D_2 D_2 D_1 - 2 D_2 D_1 D_2 + D_1 D_2 D_2).$$

Combinatorial Side Extension – Rational Dyck Paths

Definition (Rational Dyck path)

An (m, n)-Dyck path is a lattice paths from $(0, 0)$ to (m, n) which always remains weakly above the main diagonal $y = \frac{n}{m}x$.

The cells that are passed through by the main diagonal are marked as diagonal cells.

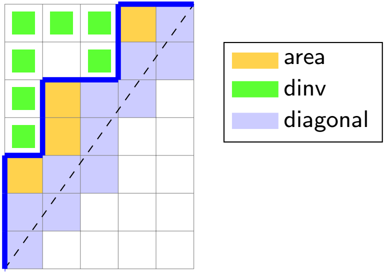

Figure 6: A rational Dyck path

Rational Dyck Paths

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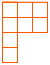
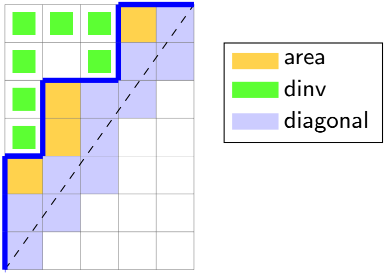
$\lambda(\Pi) = (3, 3, 1, 1)$,  .



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Rational Dyck Paths

Definition (pdinv)

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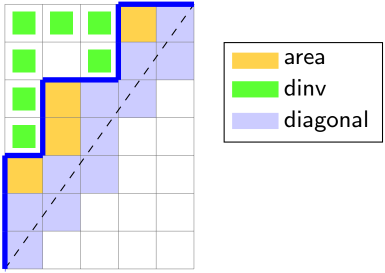

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Rational Parking Functions

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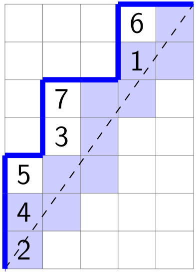
- rank of a cell is $\mathrm{rank}(x, y) = my - nx$,
- word σ: reading cars from highest → lowest rank. $\sigma(\mathrm{PF}) = 7563412$.
- $\mathrm{ides}(\sigma) = \{i \in \sigma : i + 1 \leftarrow i\}$, $\mathrm{pides}(\sigma)$ is the composition set of $\mathrm{ides}(\sigma)$. $\mathrm{ides}(\mathrm{PF}) = \{2, 4, 6\}$ and $\mathrm{pides}(\mathrm{PF}) = \{2, 2, 2, 1\}$.



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  $\sigma(\mathrm{PF}) = 7563412$.
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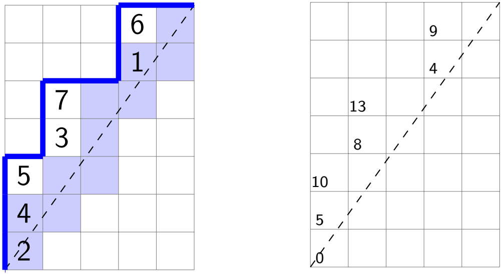

Figure 7: A $(5, 7)$-parking function and the ranks of its cars

Rational Parking Functions

Definition (tdinv)

$$\mathrm{tdinv}(\mathrm{PF}) = \sum_{\text{cars } i<j} \chi(\mathrm{rank}(i) < \mathrm{rank}(j) < \mathrm{rank}(i) + m).$$

In Figure 7, the pairs of cars contributing to tdinv are $(1,3)$, $(1,4)$, $(3,5)$, $(3,6)$, $(4,6)$, $(5,7)$ and $(6,7)$.

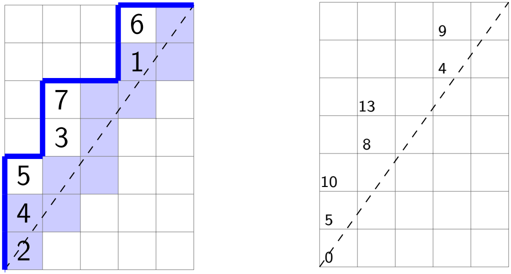

Figure 7: A $(5,7)$-parking function and the ranks of its cars

Rational Parking Functions

Leven and Hicks gave a simplified formula for the dinv of a PF. Set $\frac{0}{0} = 0$ and $\frac{x}{0} = \infty$ for all $x \neq 0$, then

Definition ($\mathrm{dinvcorr}$)

$$\begin{aligned}\mathrm{dinvcorr}(\Pi) &= \sum_{c\in\lambda(\Pi)} \chi\left(\frac{arm(c)+1}{leg(c)+1} \leq \frac{m}{n} < \frac{arm(c)}{leg(c)}\right)\\ &\quad - \sum_{c\in\lambda(\Pi)} \chi\left(\frac{arm(c)}{leg(c)} \leq \frac{m}{n} < \frac{arm(c)+1}{leg(c)+1}\right).\end{aligned}$$

Definition ($\mathrm{dinv}(\mathrm{PF})$)

Let PF be any (m,n)-parking function with underlying Dyck path Π, then

$$\mathrm{dinv}(\mathrm{PF}) = \mathrm{tdinv}(\mathrm{PF}) + \mathrm{dinvcorr}(\Pi).$$

Rational Parking Functions

- If $n > m$ then
$$\mathrm{dinv}(\mathrm{PF}) = \mathrm{tdinv}(\mathrm{PF}) - \sum_{c \in \lambda(\Pi)} \chi\left(\frac{arm(c)}{leg(c)} \leq \frac{m}{n} < \frac{arm(c)+1}{leg(c)+1}\right).$$

- If $n = m$ then $\mathrm{dinv}(\mathrm{PF}) = \mathrm{tdinv}(\mathrm{PF})$.

- Finally, if $n < m$ then
$$\mathrm{dinv}(\mathrm{PF}) = \mathrm{tdinv}(\mathrm{PF}) + \sum_{c \in \lambda(\Pi)} \chi\left(\frac{arm(c)+1}{leg(c)+1} \leq \frac{m}{n} < \frac{arm(c)}{leg(c)}\right).$$

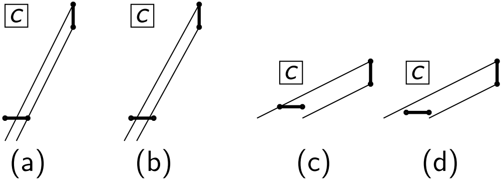

Figure 8: Types of cells that contribute to $\mathrm{dinvcorr}$

Rational Parking Functions

Definition (*ret*)

The *ret* of a (km, kn)-parking function PF is the smallest positive i such that the supporting path of PF goes through the point (im, in).

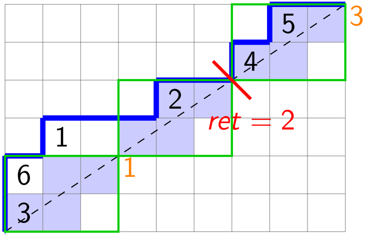

Figure 9: The *ret* of a $(9, 6)$-parking function

Extension of Shuffle Conjecture

In 2012, Hikita defined the Hikita polynomial to extend the combinatorial side of the shuffle conjecture to rational parking functions:

$$\mathrm{H}_{m,n}[X;q,t] = \sum_{\mathrm{PF}\in\mathcal{PF}_{m,n}} t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{dinv}(\mathrm{PF})} F_{\mathrm{ides}(\mathrm{PF})}[X].$$

Then the classical shuffle conjecture of HHLRU can be restated as follows.

Conjecture (Haglund-Haiman-Loehr-Remmel-Ulyanov)

For all $n \geq 0$,

$$\nabla e_n = \mathrm{H}_{n+1,n}[X;q,t].$$

Rational Shuffle Conjecture

In 2013, Gorsky and Negut introduced the operator $\mathrm{Q}_{m,n}$ and give a symmetric function expression for each coprime pair (m, n) which conjecturally coincides with $\mathrm{H}_{m,n}[X; q, t]$.

Conjecture (Gorsky-Negut)

For all pairs of coprime positive integers (m, n)*, we have*

$$\mathrm{Q}_{m,n}(-1)^n = \mathrm{H}_{m,n}[X; q, t].$$

Rational Shuffle Conjecture

In 2015, Garsia, Leven, Wallach and Xin extended the conjecture of Gorsky and Negut to any pair of integers (km, kn):

Conjecture (Garsia, Leven, Wallach and Xin)

For all pairs of coprime positive integers (m, n) *and any positive integer* k*, we have*

$$\mathrm{Q}_{km,kn}(-1)^{kn} = \sum_{\mathrm{PF} \in \mathcal{PF}_{km,kn}} [ret(\mathrm{PF})]_{\frac{1}{t}} t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{dinv}(\mathrm{PF})} F_{\mathrm{ides}(\mathrm{PF})}[X],$$

Rational Shuffle Conjecture – Solved

In 2015, Carlson and Mellit proved the Classical Shuffle Conjecture that

$$\nabla e_n = \mathrm{H}_{n+1,n}[X; q, t] = \sum_{\mathrm{PF} \in \mathcal{PF}_{n+1,n}} t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{dinv}(\mathrm{PF})} F_{\mathrm{ides}(\mathrm{PF})}[X].$$

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In April 2016, Mellit proved the Rational Shuffle Conjecture that

$$\mathrm{Q}_{km,kn}(-1)^{kn} = \sum_{\mathrm{PF}\in\mathcal{PF}_{km,kn}} [ret(\mathrm{PF})]_{\frac{1}{t}} t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{dinv}(\mathrm{PF})} F_{\mathrm{ides}(\mathrm{PF})}[X].$$

The real problem for the ring of diagonal harmonics and the $Q_{m,n}$ operators Find the Schur function expansions.

The real problem is to find the Schur function $(\{s_\lambda\})$ expansion of ∇e_n.

Similarly, we want to find the Schur function expansion of $Q_{m,n}(-1)^n$.

Schur Basis Expansion of Rational Shuffle Conjecture

$[n]_{q,t}$ is the q,t-analogue of an integer that

$$[n]_{q,t} = \frac{q^n - t^n}{q - t} = q^{n-1} + q^{n-2}t + \cdots + t^{n-1}.$$

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In 2014, Leven worked out the Schur basis Expansion for both sides of the rational shuffle conjecture when $n = 2$ and $m = 2$ that

Theorem

For any $k \geq 0$,

$$\mathrm{Q}_{2k+1,2}1 = \mathrm{H}_{2k+1,2}[X; q, t] = [k]_{q,t}s_2 + [k+1]_{q,t}s_{1,1}$$

and

$$\mathrm{Q}_{2,2k+1}1 = \mathrm{H}_{2,2k+1}[X; q, t] = \sum_{r=0}^{k}[k+1-r]_{q,t}s_{2^r1^{2k+1-2r}}.$$

Schur Basis Expansion of Rational Shuffle Conjecture

Now from the extended rational shuffle conjecture of Garsia, Leven, Wallach and Xin that

$$\mathrm{Q}_{km,kn}(-1)^{kn} = \sum_{\mathrm{PF}\in\mathcal{PF}_{km,kn}} [ret(\mathrm{PF})]_{\frac{1}{t}} t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{dinv}(\mathrm{PF})} F_{\mathrm{ides}(\mathrm{PF})}[X],$$

we have worked out that

$$\mathrm{Q}_{2k,2}1 = \mathrm{H}_{2k,2}[X;q,t] = ([k]_{q,t}+[k-1]_{q,t})s_2+([k+1]_{q,t}+[k]_{q,t})s_{1,1}$$

and

$$\mathrm{Q}_{2,2k}1 = \mathrm{H}_{2,2k}[X;q,t] = \sum_{r=0}^{k}([k+1-r]_{q,t}+[k-r]_{q,t})s_{2^r1^{2k+1-2r}}.$$

Schur Basis Expansion of Rational Shuffle Conjecture

- Problem: the Schur basis($\{s_\lambda\}$) Expansion of both sides.

Schur Basis Expansion of Rational Shuffle Conjecture

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- Our main result is the Schur expansion for $(m, 3)$ case and some partial results about $(3, n)$ case.

Schur Basis Expansion of Rational Shuffle Conjecture

- Problem: the Schur basis($\{s_\lambda\}$) Expansion of both sides.
- Our main result is the Schur expansion for $(m, 3)$ case and some partial results about $(3, n)$ case.
- We begin with the observation of $\mathrm{Q}_{m,3}(-1)$. We take $m = 3k + 1$ for an example.

Coefficients of s_λ in $Q_{3k+1,3}(-1)$

s_λ $Q_{3k+1,3}(-1)$	s_3	s_{21}	s_{1^3}
$Q_{1,3}(-1)$	0	0	$[1]_{q,t}$
$Q_{4,3}(-1)$	$[1]_{q,t}$	$[2]_{q,t}+[3]_{q,t}$	$[1]_{q,t}$ $+qt[4]_{q,t}$
$Q_{7,3}(-1)$	$[4]_{q,t}$ $+qt[1]_{q,t}$	$[5]_{q,t}+[6]_{q,t}$ $+qt([2]_{q,t}+[3]_{q,t})$	$[7]_{q,t}$ $+qt[4]_{q,t}$ $+(qt)^2[1]_{q,t}$
$Q_{10,3}(-1)$	$[7]_{q,t}$ $+qt[4]_{q,t}$ $+(qt)^2[1]_{q,t}$	$[8]_{q,t}+[9]_{q,t}$ $+qt([5]_{q,t}+[6]_{q,t})$ $+(qt)^2([2]_{q,t}+[3]_{q,t})$	$[10]_{q,t}$ $+qt[7]_{q,t}$ $+(qt)^2[4]_{q,t}$ $+(qt)^3[1]_{q,t}$
$Q_{13,3}(-1)$	$[10]_{q,t}$ $+qt[7]_{q,t}$ $+(qt)^2[4]_{q,t}$ $+(qt)^3[1]_{q,t}$	$[11]_{q,t}+[12]_{q,t}$ $+qt([8]_{q,t}+[9]_{q,t})$ $+(qt)^2([5]_{q,t}+[6]_{q,t})$ $+(qt)^3([2]_{q,t}+[3]_{q,t})$	$[13]_{q,t}$ $+qt[10]_{q,t}$ $+(qt)^2[7]_{q,t}$ $+(qt)^3[4]_{q,t}$ $+(qt)^4[1]_{q,t}$

Main Result

Formula for the Coefficients of Schur function expansion when $n = 3$.

Theorem

Let $[s_\lambda]_{m,n}$ be the coefficient of Schur basis s_λ in the polynomial $\mathrm{Q}_{m,n}(-1)$ and the polynomial $\mathrm{H}_{m,n}[X; q, t]$, then

(1)

$$\begin{aligned}
[s_3]_{3k+1,3} &= \sum_{i=0}^{k-1} (qt)^{k-1-i} [3i+1]_{q,t}, \\
[s_{21}]_{3k+1,3} &= \sum_{i=0}^{k-1} (qt)^{k-1-i} ([3i+2]_{q,t} + [3i+3]_{q,t}), \\
[s_{1^3}]_{3k+1,3} &= [s_3]_{3k+4,3};
\end{aligned}$$

Formula for the Coefficients of Schur Basis When $n = 3$

(2)

$$\begin{aligned}
[s_3]_{3k+2,3} &= \sum_{i=0}^{k-1} (qt)^{k-1-i} [3i+2]_{q,t}, \\
[s_{21}]_{3k+2,3} &= \sum_{i=0}^{k} (qt)^{k-1-i} ([3i]_{q,t} + [3i+1]_{q,t}), \\
[s_{1^3}]_{3k+2,3} &= [s_3]_{3k+5}\,;
\end{aligned}$$

(3)

$$
\begin{aligned}
[s_3]_{3k,3} &= \sum_{i=0}^{k-1} (qt)^{k-1-i}([3i-1]_{q,t} + [3i]_{q,t} + [3i+1]_{q,t}), \\
[s_{21}]_{3k,3} &= (qt)^{k+1}([3]_{q,t} + 2[2]_{q,t} + [1]_{q,t}) \\
&\quad + \sum_{i=1}^{k-1} (qt)^{k-1-i}([3i]_{q,t} + 2[3i+1]_{q,t} \\
&\qquad\qquad + 2[3i+2]_{q,t} + [3i+3]_{q,t}), \\
[s_{1^3}]_{3k,3} &= [s_3]_{3k+3}\,.
\end{aligned}
$$

Example: Formula for $[s_3]_{3k+1,3}$

Theorem

The coefficient of Schur basis s_3 in the polynomial $\mathrm{Q}_{3k+1,3}(-1)$ and the polynomial $\mathrm{H}_{3k+1,3}[X;q,t]$ is

$$[s_3]_{3k+1,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i} [3i+1]_{q,t}$$

Symmetric Function Side

The Coefficient of Schur Basis s_3 in the Polynomial $\mathrm{Q}_{3k+1,3}(-1)$

We need the following lemma from Bergeron, Garsia, Leven and Xin to prove the symmetric function side of the theorem.

Lemma

For any positive m, n*,*

$$\nabla \mathrm{Q}_{m,n} \nabla^{-1} = \mathrm{Q}_{m+n,n}.$$

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Lemma

For any positive m, n*,*

$$\nabla \mathrm{Q}_{m,n} \nabla^{-1} = \mathrm{Q}_{m+n,n}.$$

From the lemma, we can get a recursion for $\mathrm{Q}_{m,n}$ operator that

$$\mathrm{Q}_{m+n,n}(-1)^n = \nabla \mathrm{Q}_{m,n} \nabla^{-1}(-1)^n = \nabla \mathrm{Q}_{m,n}(-1)^n, \quad \textit{and}$$

$$\mathrm{Q}_{3(k+1)+1,3}(-1)^n = \nabla \mathrm{Q}_{3k+1,3} \nabla^{-1}(-1)^n = \nabla \mathrm{Q}_{3k+1,3}(-1)^n.$$

Algebraic Proof

We first apply the operator ∇ to the Schur basis s_3, s_{21} and s_{1^3}:

$$\begin{aligned}\nabla s_3 &= (qt)^2 s_{21} + (qt)^2 [2]_{q,t} s_{1^3},\\ \nabla s_{21} &= (qt)[2]_{q,t} s_{21} - (qt)[3]_{q,t} s_{1^3},\\ \nabla s_{1^3} &= s_3 + ([2]_{q,t} + [3]_{q,t}) s_{21} + (qt + [4]_{q,t}) s_{1^3}.\end{aligned}$$

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\nabla s_{1^3} &= s_3 + ([2]_{q,t} + [3]_{q,t}) s_{21} + (qt + [4]_{q,t}) s_{1^3}.
\end{aligned}$$

Then we can apply ∇ to the polynomial $\mathrm{Q}_{3k+1,3}(-1)$.

$$\begin{aligned}
& \nabla \mathrm{Q}_{3k+1,3}(-1) \\
= \;& \nabla([s_3]_{3k+1,3} s_3 + [s_{21}]_{3k+1,3} s_{21} + [s_{1^3}]_{3k+1,3} s_{1^3}) \\
= \;& [s_3]_{3k+1,3} \nabla s_3 + [s_{21}]_{3k+1,3} \nabla s_{21} + [s_{1^3}]_{3k+1,3} \nabla s_{1^3} \\
= \;& [s_{1^3}]_{3k+1,3} s_3 \\
& + \left[(qt)^2 [s_3]_{3k+1,3} - qt[2]_{q,t} [s_{21}]_{3k+1,3} + ([2]_{q,t} + [3]_{q,t}) [s_{1^3}]_{3k+1,3}\right] \\
& + \left[(qt)^2 [2]_{q,t} [s_3]_{3k+1,3} - qt[3]_{q,t} [s_{21}]_{3k+1,3} + (qt + [4]_{q,t}) [s_{1^3}]_{3k+1,\right. \\
= \;& [s_3]_{3k+4,3} s_3 + [s_{21}]_{3k+4,3} s_{21} + [s_{1^3}]_{3k+4,3} s_{1^3},
\end{aligned}$$

Algebraic Proof

and the recursion from $[s_\lambda]_{3k+1,3}$ to $[s_\lambda]_{3k+4,3}$ is clear that

$$[s_3]_{3k+4,3} = [s_{1^3}]_{3k+1,3},$$

$$[s_{21}]_{3k+4,3} = (qt)^2[s_3]_{3k+1,3} - qt[2]_{q,t}[s_{21}]_{3k+1,3} + ([2]_{q,t} + [3]_{q,t})[s_{1^3}]_{3k+1,3},$$

$$[s_{1^3}]_{3k+4,3} = (qt)^2[2]_{q,t}[s_3]_{3k+1,3} - qt[3]_{q,t}[s_{21}]_{3k+1,3} + (qt + [4]_{q,t})[s_{1^3}]_{3k+1}$$

Combinatorial Side – From F_α to s_λ

Hikita(2012) proved that Hikita polynomials $\mathrm{H}_{m,n}[X;q,t]$ are symmetric (in X) for any coprime m, n.

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Theorem (Garsia and Remmel)

Suppose that $P(X)$ is a symmetric function which is homogeneous of degree n and $P(X) = \sum_{\alpha \Vdash n} a_\alpha F_\alpha(X)$, Then

$$P(X) = \sum_{\alpha \Vdash n} a_\alpha s_{\tilde{\alpha}}(X).$$

Here $\tilde{\alpha}$ is the composition set of α, and $s_\alpha(X) = \frac{\Delta_\alpha(X)}{\Delta(X)}$.

This allows us to transform $\mathrm{H}_{m,n}[X;q,t]$ into Schur function expansion.

From F_α to s_λ — Straightening

- Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition of n. Suppose that for some i, $\alpha_i < \alpha_{i+1}$ (i.e. α is not a partition). Then $s_\alpha = -s_{(\alpha_1,\ldots,\alpha_{i+1}-1,\alpha_i+1,\ldots,\alpha_k)}$. This action is called straightening.

From F_α to s_λ — Straightening

- Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition of n. Suppose that for some i, $\alpha_i < \alpha_{i+1}$ (i.e. α is not a partition). Then $s_\alpha = -s_{(\alpha_1,\ldots,\alpha_{i+1}-1,\alpha_i+1,\ldots,\alpha_k)}$. This action is called straightening.
- Repeatedly applying this procedure will eventually yield a partition or a composition α' such that $\alpha'_j = \alpha'_{j+1} - 1$ for some j. In the latter case, the straightening action yields $s_{\alpha'} = -s_{\alpha'}$, hence $s_{\alpha'} = 0$.
- Ex. $s_{2,3,1} = -s_{3-1,2+1,1} = -s_{2,3,1} = 0.$
- Ex. $s_{1,3,1} = -s_{3-1,1+1,1} = -s_{2,2,1}.$

Notation for the Coeff of s_λ

- We define

$$
\begin{aligned}
[s_\sigma]_{m,n}(q,t) &= \sum_{\mathrm{PF}\in\mathcal{PF}_{m,n,s_\sigma}} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} \frac{s_{\mathrm{pides(PF)}}}{s_\sigma} \\
&= \sum_{\alpha \text{ straightened to } \sigma} h_{m,n,\mathrm{pides}\ \alpha}(q,t) \frac{s_\alpha}{s_\sigma}.
\end{aligned}
$$

- Then naturally $[s_\sigma]_{m,n}(q,t)$ is the coefficient of s_σ in $\mathrm{H}_{m,n}[X;q,t]$, i.e.

$$\mathrm{H}_{m,n}[X;q,t] = \sum_{\sigma\vdash n} [s_\sigma]_{m,n}(q,t) s_\sigma.$$

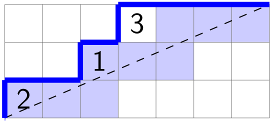
Notation for the Coeff of s_λ

- Recall that the combinatorial side is the Hikita polynomial:

$$\mathrm{H}_{m,3}[X;q,t] = \sum_{\mathrm{PF}\in\mathcal{PF}_{m,3}} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} F_{\mathrm{ides(PF)}}[X].$$

- By the action straightening, we can transform it to

$$\mathrm{H}_{m,3}[X;q,t] = \sum_{\mathrm{PF}\in\mathcal{PF}_{m,3}} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} s_{\mathrm{pides(PF)}}[X].$$



Combinatorial Side Proof

Any parking function $\mathrm{PF} \in \mathcal{PF}_{m,3}$ has 3 rows, thus has only 3 cars: $1, 2, 3$. So the word $\sigma(\mathrm{PF})$ can be any permutation $\sigma \in \mathcal{S}_3$. Table 1 shows the s_{pides} contribution of the 6 permutations in $\mathcal{S}_3$.

$\sigma \in \mathcal{S}_3$	s_{pides}
123	s_3
132	s_{21}
213	$s_{12} = 0$
231	s_{21}
312	$s_{12} = 0$
321	s_{1^3}

Table 1: Coefficients of s_λ in $\mathrm{Q}_{3k+1,3}(-1)$

Since there are only 3 partitions of 3: $\{3, 21, 1^3\}$, the Hikita polynomial of $(m, 3)$ case is

$$\mathrm{H}_{m,3}[X; q, t] = [s_3]_{m,3} s_3 + [s_{21}]_{m,3} s_{21} + [s_{1^3}]_{m,3} s_{1^3}.$$

Combinatorial Side Proof

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Table 1: Coefficients of s_λ in $Q_{3k+1,3}(-1)$

From the table we can see that

- $[s_3]_{m,3} = h_{m,3,\text{word } 123}$,
- $[s_{21}]_{m,3} = h_{m,3,\text{word } 132} + h_{m,3,\text{word } 231}$,
- $[s_{1^3}]_{m,3} = h_{m,3,\text{word } 321}$,

The combinatorics of $[s_3]_{3k+1,3}$

We take $[s_3]_{3k+1,3}$ as an example. We will construct

$$[s_3]_{3k+1,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i} [3i+1]_{q,t}.$$

Since $[s_3]_{m,3} = h_{m,3,\text{word } 123}$, we are looking at the set of parking functions in $\mathcal{PF}_{m,3,\text{word } 123}$.

This set $\mathcal{PF}_{m,3,\text{word } 123}$ of parking functions can be obtained by adding cars $1, 2, 3$ in a rank-decreasing way to a $m \times 3$ Dyck path, and smaller cars can't be put on top of bigger cars, so we have one $\mathrm{PF} \in \mathcal{PF}_{m,3,\text{word123}}$ on each $m \times 3$ Dyck path with no consecutive u, u steps.

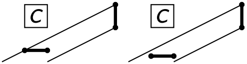
The combinatorics of $[s_3]_{3k+1,3}$

Recall that tdinv is defined as

$$\mathrm{tdinv}(\mathrm{PF}) = \sum_{\mathrm{cars}\ i<j} \chi(\mathrm{rank}(i) < \mathrm{rank}(j) < \mathrm{rank}(i) + m).$$

Since the word is 123, we have $\mathrm{rank}(1) > \mathrm{rank}(2) > \mathrm{rank}(3)$, so there will always be no tdinv for $\mathrm{PF} \in \mathcal{PF}_{m,3,\mathrm{word}123}$. Since $m = 3k+1 > n = 3$ for $k \geq 1$, the dinv correction is of the third type. We have

$$\mathrm{dinv}(\mathrm{PF}) = \mathrm{dinvcorr}(\mathrm{PF}) = \sum_{c \in \lambda(\Pi)} \chi\left(\frac{arm(c)+1}{leg(c)+1} \leq \frac{m}{n} < \frac{arm(c)}{leg(c)} \right).$$



(c) (d)

The combinatorics of $[s_3]_{3k+1,3}$

The partition $\lambda(\Pi)$ correspond with the Dyck path Π of $\mathrm{PF} \in \mathcal{PF}_{m,3}$ is at most of hight 2, so the leg of cells in $\lambda(\Pi)$ can be either 0 or 1. Taking Figure 9 for reference, we have

(a) Cells in $\lambda(\Pi)$ with $\mathrm{leg} = 0$ and $1 < \mathrm{arm} < k$ contribute 1 to dinv correction, marked ◯ in Figure 9,

(b) Cells in $\lambda(\Pi)$ with $\mathrm{leg} = 1$ and $k < \mathrm{arm} < 2k - 1$ contribute 1 to dinv correction, marked △ in Figure 9.

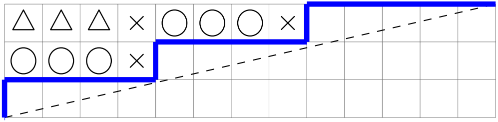


Figure 9: The dinv correction of a $(3k+1) \times 3$ Dyck path

The combinatorics of $[s_3]_{3k+1,3}$

We can count dinv correction and area according to the partition $\lambda(\Pi)$ of the path Π. Each path Π corresponds with a partition $\lambda = (\lambda_1, \lambda_2) \subseteq \lambda_0 = (2k, k)$. Let $\mathrm{dinvcorr}(\lambda(\Pi)) = \mathrm{dinvcorr}(\Pi)$ and $\mathrm{area}(\lambda(\Pi)) = \mathrm{area}(\Pi)$, then

$$\mathrm{area}(\Pi) = 2k - \lambda_1 - \lambda_2,$$

and we can also write the formula for dinv correction:

$$\mathrm{dinvcorr}(\lambda) = \begin{cases} \lambda_1 - 2 & \text{if } \lambda_2 \geq 1, 1 \leq \lambda_1 - \lambda_2 \leq k, \text{ and } \lambda_1 \leq k \\ 2\lambda_1 - k - 3 & \text{if } \lambda_2 \geq 1, 1 \leq \lambda_1 - \lambda_2 \leq k, \text{ and } \lambda_1 \geq k+1 \\ 2\lambda_2 + k - 2 & \text{if } \lambda_2 \geq 1 \text{ and } \lambda_1 - \lambda_2 \geq k+1 \end{cases}.$$

The combinatorics of $[s_3]_{3k+1,3}$

Now for $[s_3]_{3k+1,3} = \sum_{i=0}^{k-1}(qt)^{k-1-i}[3i+1]_{q,t}$, we construct each term $(qt)^{k-1-i}[3i+1]_{q,t}$ as a sequence of parking functions.

For each i, we have 3 branches of partitions(or parking functions):

$$\begin{aligned}
\Lambda_1 &= \{(k+i+1,k),(k+i,k-1),\ldots,(k+1,k-i)\},\\
\Lambda_2 &= \{(2k,i),(2k-1,i-1),\ldots,(2k+1-i,1)\},\\
\Lambda_3 &= \{(k+1,i+1),(k,i+1),\ldots,(i+2,i+1)\}.
\end{aligned}$$

- The branch Λ_1 contains λ's such that $\lambda_1 - \lambda_2 = i+1 \leq k$ with $\lambda_2 \geq i+1$,
- the branch Λ_2 contains all λ's such that $\lambda_1 - \lambda_2 = 2k - i > k$, and
- the branch Λ_3 contains λ's such that $\lambda_2 = i+1$ and $\lambda_1 - \lambda_2 \leq k - i$.

The combinatorics of $[s_3]_{3k+1,3}$

$|\Lambda_1| = |\Lambda_2| + 1$, and the last partition of Λ_1 is the same as the first partition in Λ_3. So as shown in Figure 10, the construction begin with alternatively taking partitions from Λ_1 and Λ_2, ending with the last partition of Λ_1. Then continue the chain by taking partitions in Λ_3 and end the chain with the last partition $(k - i + 1, k - i)$ in Λ_3.

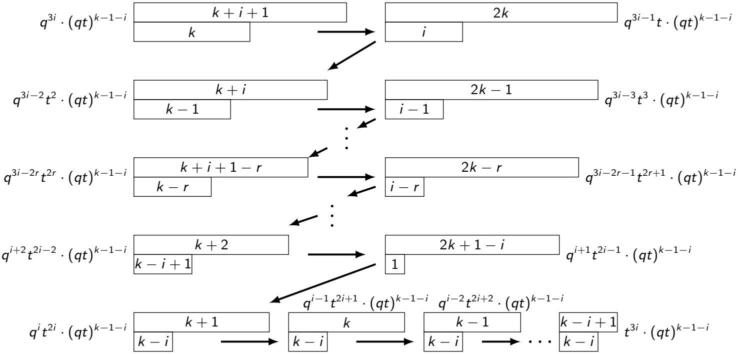


Figure 10: The construction of $(qt)^{k-1-i}[3i + 1]_{q,t}$

The combinatorics of $[s_3]_{3k+1,3}$

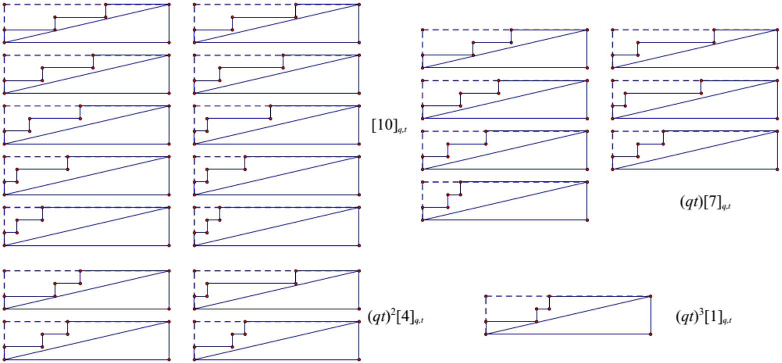
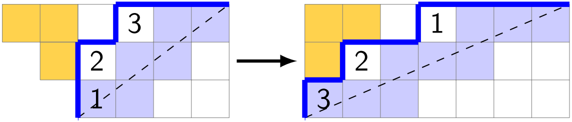


Figure 11: The construction of $[s_3]_{13,3}$

The combinatorics of $[s_{1^3}]_{3k+1,3}$

- $[s_{1^3}]_{m-3,3} = [s_3]_{m,3}$. Bijection:

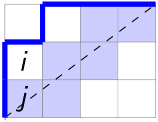


- $[s_{21}]_{m,3}$ is a construction problem similar to $[s_3]_{m,3}$.
- For the case $m = 3$, we have several results about $[s_\lambda]_{3,n}$. Every equation about $[s_\lambda]_{3,n}$ implies a bijection about parking functions.

Remark about pides in $(3, n)$ Case

Remark

Let $i < j$ be two cars in the parking function. If i appears to the left of j in the diagonal word, then the cars i, j must be in different columns.



Remark

The elements in the pides *of a parking function* $\mathrm{PF} \in \mathcal{PF}_{m,n}$ *is at most* m*.*

So in $(3, n)$ case, the λ in $[s_\lambda]_{3,n}$ can only be of form $3^a 2^b 1^c$ with $3a + 2b + c = n$.

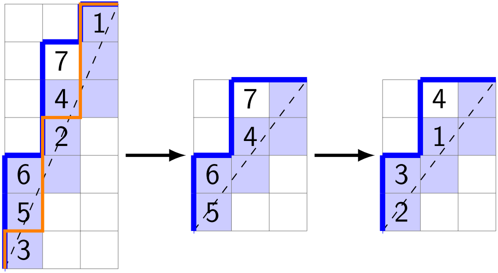
Coefficients of s_λ in $\mathrm{Q}_{3,3k+1}(-1)^{3k+1}$

$$\mathrm{Q}_{3,4,1} = s_{31} + [2]_{q,t} s_{2^2} + ([3]_{q,t} + [2]_{q,t}) s_{21^2} + ([4]_{q,t} + (qt)[1]_{q,t}) s_{1^4}$$

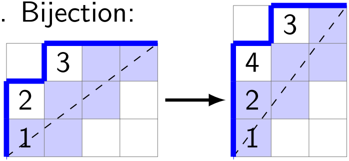
$$\begin{aligned}
&\mathrm{Q}_{3,7,-1} = \\
&s_{3^21} + [2]_{q,t} s_{32^2} + ([3]_{q,t} + [2]_{q,t}) s_{321^2} + ([4]_{q,t} + (qt)[1]_{q,t}) s_{31^4} \\
&\qquad + ([4]_{q,t} + [3]_{q,t} + (qt)[1]_{q,t}) s_{2^31} \\
&\qquad + ([5]_{q,t} + [4]_{q,t} + [3]_{q,t} + (qt)[2]_{q,t}) s_{2^21^3} \\
&\qquad + ([6]_{q,t} + [5]_{q,t} + (qt)([3]_{q,t} + [2]_{q,t})) s_{21^5} \\
&\qquad + ([7]_{q,t} + [4]_{q,t} + [1]_{q,t}) s_{1^7}
\end{aligned}$$

Combinatorial Results about $[s_{\lambda}]_{3,n}$

- $[s_{3^{a+1}2^{b}1^{c}}]_{3,n} = [s_{3^{a}2^{b}1^{c}}]_{3,n-3}$. Bijection:

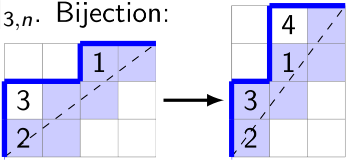


- $[s_{1^{3}}]_{n,3} = [s_{1^{n}}]_{3,n}$. Bijection:



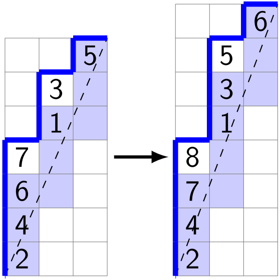
Combinatorial Results about $[s_\lambda]_{3,n}$

- $[s_{21}]_{n,3} = [s_{21^{n-2}}]_{3,n}$. Bijection:



- Straitening action: $\mathrm{pides}\{\cdots 1, 3 \cdots\} \Rightarrow \mathrm{pides}\{\cdots 2, 2 \cdots\}$ for $\mathcal{PF}_{3,n}$ is clear – an involution whose fixed points are the coefficients of $[s_{2^a1^b}]_{3,n}$.

- $[s_{2^a1^b}]_{3,n} = [s_{2^b1^a}]_{3,3(a+b)-n}$. Bijection:



Combinatorial Results about $[s_\lambda]_{3,n}$

Conjecture

Let $a < b$, then

$$[s_{2^a1^b}]_{3,n} = \sum_{i=0}^{a}[b+i]_{q,t} + (qt)[s_{2^a1^{b-3}}]_{3,n-3}.$$

We verified this formula by Maple for $n < 27$. If this conjecture is true, then we have solved the Schur function expansion in the $(3, n)$ case.

Combinatorial Results about $[s_\lambda]_{m,n}$

Theorem

For all $m, n > 0$ *and* $\lambda' \vdash (n - am)$*,*
(a) $[s_{1^n}]_{m-n,n} = [s_n]_{m,n}$, **(b)** $[s_{m^a\lambda'}]_{m,n} = [s_{\lambda'}]_{m,n-am}$,
(c) $[s_{1^n}]_{m,n} = [s_{1^m}]_{n,m}$, **(d)** $[s_{k1^{n-k}}]_{m,n} = [s_{k1^{m-k}}]_{n,m}$.

Conjecture

$$[s_{(m-1)^{\alpha_{m-1}}(m-2)^{\alpha_{m-2}}\cdots 1^{\alpha_1}}] = [s_{(m-1)^{\alpha_1}(m-2)^{\alpha_2}\cdots 1^{\alpha_{m-1}}}].$$

Thank You!