Combinatorial Problems in Dyck paths and Parking Functions

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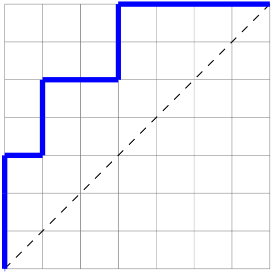
$n \times n$ Dyck Paths

Definition (Dyck path)

An $n \times n$ Dyck path is a lattice path from $(0,0)$ to (n,n) consisting of east and north steps which stays above the diagonal $y = x$.

The set of $n \times n$ Dyck paths is denoted $\mathcal{D}_n$, and $|\mathcal{D}_n| = C_n = \frac{1}{n+1}\binom{2n}{n}$.



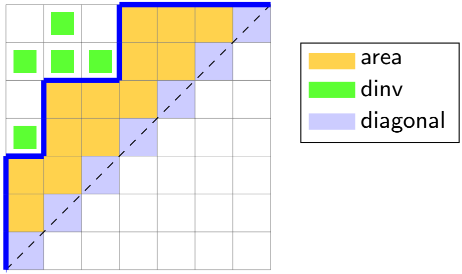
A $(7,7)$-Dyck path

Area of a Dyck Path

Definition (area)

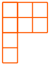
The number of full cells between an (n, n)-Dyck path Π and the main diagonal is denoted $area(\Pi)$.

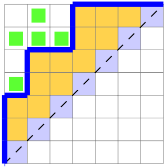
The number of full cells above the Dyck path Π is denoted $coarea(\Pi)$.

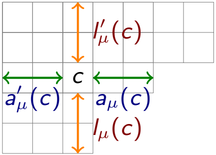


A $(7, 7)$-Dyck path

Partition of a Dyck Path

The collection of cells above Π forms an the Ferrers diagram of a partition $\lambda(\Pi)$. Ex. $\lambda(\Pi) = (3, 3, 1, 1)$, .





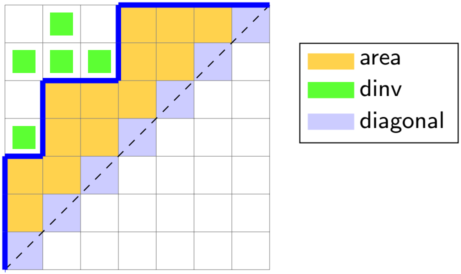
$$\begin{aligned} a_\mu(c) &= \text{arm of } c, \\ a'_\mu(c) &= \text{coarm of } c, \\ l_\mu(c) &= \text{leg of } c, \\ l'_\mu(c) &= \text{coleg of } c. \end{aligned}$$

Dinv of a Dyck Path

Definition (dinv)

The dinv of an (n, n)-Dyck path Π is given by

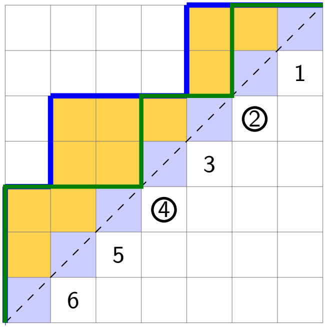
$$\mathrm{dinv}(\Pi) = \sum_{c \in \lambda(\Pi)} \chi\left(\frac{arm(c)}{leg(c)+1} \leq 1 < \frac{arm(c)+1}{leg(c)}\right).$$



A $(7, 7)$-Dyck path

Bounce of a Dyck Path

$bounce(\Pi) = 4 + 2 = 6.$



A $(7,7)$-Dyck path

Polynomials about $n \times n$ Dyck paths

t, q, s-Catalan polynomial

$$C_n(t,q,s) = \sum_{\Pi \in \mathcal{D}_n} t^{area(\Pi)} q^{dinv(\Pi)} s^{bounce(\Pi)}.$$

- $C_n(t,q,s)$ is not symmetric in t, q, s.
- $C_n(1,1,1) = C_n = \frac{1}{n+1}\binom{2n}{n}$.
- $C_n(q,1,1) = C_n(1,q,1) = C_n(1,1,q)$: q-Catalan number.
- Let $T_n(q) = \sum_{\text{T rooted planar tree of } n+1 \text{ vertices}} q^{inv(T)}$, then

$$C_n(q,1,1) = T_n(q)$$

t, q, s-Catalan polynomial

$$C_n(t,q,s) = \sum_{\Pi \in \mathcal{D}_n} t^{area(\Pi)} q^{dinv(\Pi)} s^{bounce(\Pi)}.$$

- $C_n(t,q,1) = C_n(q,1,t)$ is symmetric in q and t, called q, t-Catalan number.
- Haglund's Zeta map: dinv$\rightarrow$area, area$\rightarrow$bounce.
- No direct combinatorial approach on dinv$\longleftrightarrow$area.

Parking Functions

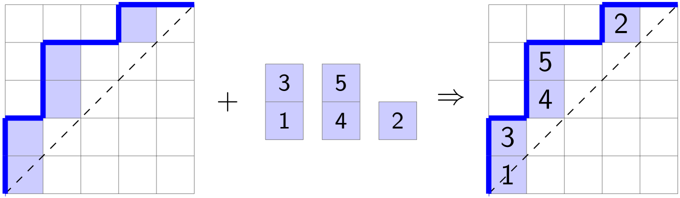
Definition (Parking Functions – Richard Stanley)

Let $\alpha = (a_1, \ldots, a_n) \in \mathbb{P}^n$. Let $b_1 \leq \cdots \leq b_n$ be the increasing rearrangement of α. Then α is a parking function iff $b_i \leq i$.

Garsia's construction of parking functions – labeling Dyck paths.

We can get an $n \times n$ parking function by labeling the cells east of and adjacent to a north step of a Dyck path with numbers $1, \ldots, n$.

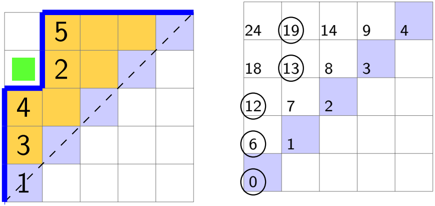
The set of $n \times n$ parking functions is denoted $\mathcal{PF}_n$.



The construction of a parking function

Statistics of an (n, n)-PF

- $\mathrm{area(PF)} = \mathrm{area}(\Pi(\mathrm{PF})) = 8$,
- rank of a cell is $\mathrm{rank}(x, y) = (n+1)y - nx$,
- $\mathrm{dinv(PF)} = \sum_{cars\ i<j} \chi(rank(i) < rank(j) \leq rank(i) + n) = 0$,



A $(5, 5)$-Parking Function

q, t-Polynomials about $n \times n$ Parking Functions

$PF_n(q, t)$

$$PF_n(q,t) = \sum_{\mathrm{PF} \in \mathcal{PF}_n} t^{area(\mathrm{PF})} q^{dinv(\mathrm{PF})}.$$

- $PF_n(1,1) = (n+1)^{(n-1)}$.
- $PF_n(q,1) = PF_n(1,q) = P_n(q)$ satisfy the recursion

$$P_{n+1}(q) = \sum_{i=0}^{n} \binom{n}{i} [i+1]_q P_i(q) P_{n-i}(q).$$

- Let $LT_n(q) = \sum_{\text{T rooted labeled tree of } n+1 \text{ vertices}} q^{inv(T)}$, then

$$P_n(q) = LT_n(q)$$

q, t-Polynomials about $n \times n$ Parking Functions

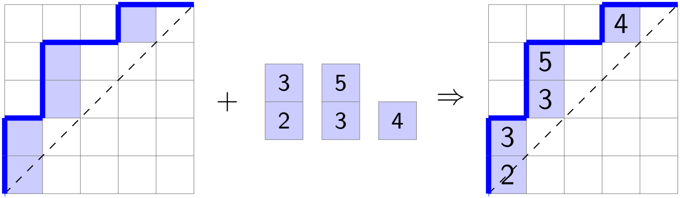
$PF_n(q,t)$

$$PF_n(q,t) = \sum_{\mathrm{PF} \in \mathcal{PF}_n} t^{area(\mathrm{PF})} q^{dinv(\mathrm{PF})}.$$

- $PF_n(q,t) = \langle \nabla e_n, p_1^n \rangle$ is symmetric in t, q as a consequence of the Shuffle conjecture.
- There is no elementary combinatorial proof that $PF_n(t,q) = PF_n(q,t)$.
- There are well studied bijections between parking functions and rooted labeled trees.

(n, n)-Labeled Dyck paths

- We can get an $n \times n$ labeled Dyck path by labeling the cells east of and adjacent to a north step of a Dyck path with numbers in (P).
- The set of $n \times n$ labeled Dyck paths is denoted $\mathcal{LD}_n$.
- Weight of $P \in \mathcal{LD}_n$ is $t^{\mathrm{area}(P)} q^{\mathrm{dinv}(P)} X^P$.



The construction of a labeled Dyck path with weight $t^5 q^3 x_2 x_3^2 x_4 x_5$.

Symmetric Functions about $n \times n$ Parking Functions

For any given path $\Pi \in (D)_n$, the LLT polynomial corresponding to Π is

LLT Polynomials

$$LLT_{\Pi}[X;q,t] = \sum_{P \text{ labeled Dyck path on } \Pi} t^{\mathrm{area}(P)} q^{\mathrm{dinv}(P)} X^P.$$

- LLT_{Π} is symmetric and Schur positive in $X = \{x_1, x_2, \ldots\}$, proved by Algebraic Geometry.
- There is currently even NO proof of symmetry combinatorially.
- We have a combinatorial translation of the LLT symmetry.

Symmetric Functions about $n \times n$ Parking Functions

Further, we have the Hikita polynomimal,

Hikita Polynomials

$$H_n[X; q, t] = \sum_{\Pi \in \mathcal{D}_n} LLT_{\Pi}.$$

- H_n is symmetric in q, t, as a consequence of the Shuffle conjecture.
- No combinatorial proof of q, t-symmetry.

Shuffle Conjecture (proved by Carlson and Mellit)

$$\nabla e_n = H_n.$$

Delta Conjecture (not proved yet)

$$\Delta_{e_k} e_n = \sum_{P \in \mathcal{LD}_n^*} t^{\mathrm{area}^-(P)} q^{\mathrm{dinv}(P)} X^P.$$

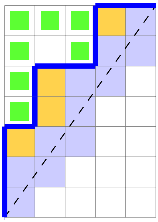
Rational Extension – Rational Dyck Paths

Definition (Rational Dyck path)

An (m, n)-Dyck path is a lattice paths from $(0, 0)$ to (m, n) which always remains weakly above the main diagonal $y = \frac{n}{m}x$.

The cells that are passed through by the main diagonal are marked as diagonal cells.

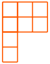
The number of (m, n)-Dyck path is $C_{m,n} = \frac{1}{m+n}\binom{m+n}{m,n}$.

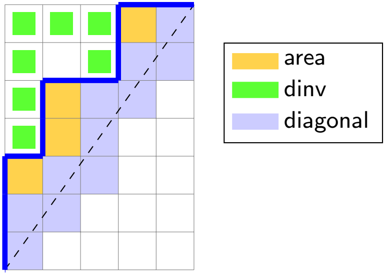


A rational Dyck path

Definition (area)

The number of full cells between an (m, n)-Dyck path Π and the main diagonal is denoted $area(\Pi)$.

The collection of cells above Π forms the Ferrers diagram of a partition $\lambda(\Pi)$. Ex. $\lambda(\Pi) = (3, 3, 1, 1)$,  .

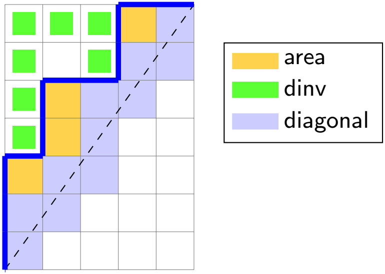


A rational Dyck path

Definition (pdinv)

The path dinv of an (m, n)-Dyck path Π is given by

$$\mathrm{pdinv}(\Pi) = \sum_{c \in \lambda(\Pi)} \chi\left(\frac{arm(c)}{leg(c)+1} \leq \frac{m}{n} < \frac{arm(c)+1}{leg(c)}\right).$$



A rational Dyck path

Polynomials about rational Dyck paths

rational q, t-Catalan number

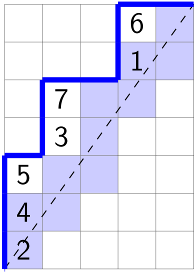
$$C_{m,n}(q,t) = \sum_{\Pi \in \mathcal{D}_{m,n}} t^{area(\Pi)} q^{pdinv(\Pi)}.$$

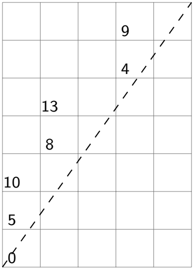
- $C_n(q,t)$ is symmetric in q, t.
- There is a Phi map analogues to Haglund's Zeta map: dinv$\rightarrow$area, area$\rightarrow$something like bounce.
- No direct combinatorial approach on dinv$\longleftrightarrow$area.

Rational Parking Functions

We label the north steps to get rational parking functions and rational labeled Dyck paths.

- $\mathrm{area}(\mathrm{PF}) = \mathrm{area}(\Pi(\mathrm{PF})) = 4$,
- rank of a cell is $\mathrm{rank}(x, y) = my - nx$,





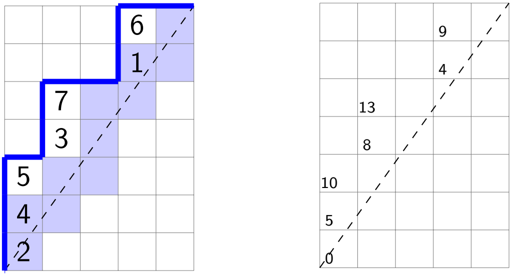
A $(5, 7)$-parking function and the ranks of its cars

Rational Parking Functions

Definition (tdinv)

$$\text{tdinv}(\text{PF}) = \sum_{\text{cars } i<j} \chi(\text{rank}(i) < \text{rank}(j) < \text{rank}(i) + m).$$

In the following example, the pairs of cars contributing to tdinv are $(1,3)$, $(1,4)$, $(3,5)$, $(3,6)$, $(4,6)$, $(5,7)$ and $(6,7)$.



A $(5,7)$-parking function and the ranks of its cars

Rational Parking Functions

Leven and Hicks gave a simplified formula for the dinv of a PF. Set $\frac{0}{0} = 0$ and $\frac{x}{0} = \infty$ for all $x \neq 0$, then

Definition (dinvcorr)

$$\begin{aligned} \mathrm{dinvcorr}(\Pi) &= \sum_{c \in \lambda(\Pi)} \chi\left(\frac{arm(c)+1}{leg(c)+1} \leq \frac{m}{n} < \frac{arm(c)}{leg(c)}\right) \\ &- \sum_{c \in \lambda(\Pi)} \chi\left(\frac{arm(c)}{leg(c)} \leq \frac{m}{n} < \frac{arm(c)+1}{leg(c)+1}\right). \end{aligned}$$

Definition (dinv(PF))

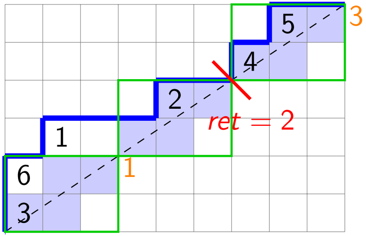
Let PF be any (m, n)-parking function with underlying Dyck path Π, then

$$\mathrm{dinv}(\mathrm{PF}) = \mathrm{tdinv}(\mathrm{PF}) + \mathrm{dinvcorr}(\Pi).$$

Rational Parking Functions

Definition (return)

The return(ret) of a (km, kn)-parking function PF is the smallest positive i such that the supporting path of PF goes through the point (im, in).



The *ret* of a $(9, 6)$-parking function

Hikita Polynomial

For coprime m, n,

$$\mathrm{H}_{km,kn}[X;q,t] = \sum_{P \in \mathcal{LD}_{km,kn}} [ret(P)]_{\frac{1}{t}} t^{\mathrm{area}(P)} q^{\mathrm{dinv}(P)} X^P.$$

$\mathrm{H}_{km,kn}[X;q,t]$ is symmetric in X, symmetric in q,t and Schur positive.

Rational Shuffle Conjecture

In 2013, Gorsky and Negut introduced the operator $\mathrm{Q}_{m,n}$ and give a symmetric function expression for each coprime pair (m, n) which conjecturally coincides with $\mathrm{H}_{m,n}[X; q, t]$. The Rational Shuffle Conjecture was proved by Mellit.

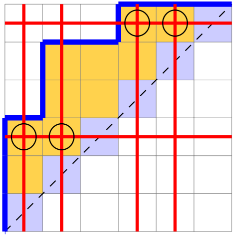
Theorem (Mellit)

For all pairs of coprime positive integers (m, n), we have

$$\mathrm{Q}_{m,n}(-1)^n = \mathrm{H}_{m,n}[X; q, t].$$

Combinatorial Problem 1 – LLT Symmetry

For an $n \times n$ Dyck path Π, we assign either a horizontal or a vertical line at each diagonal cell, and count the intersections of the lines in the area cells.

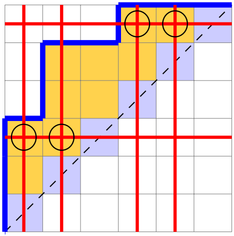


There are 2^n ways to assign the lines.

Let

$$F_{\Pi,k}(q) = \sum_{\text{assignments with k verticle}} q^{\left|\text{intersections in area}\right|}.$$

Combinatorial Problem 1 – LLT Symmetry



It is clear that $F_{\Pi,k}(1) = \binom{n}{k}$.

LLT of path Π is symmetric iff $F_{\Pi,k}(q) = F_{\Pi,n-k}(q)$ for all k.

Problem: We want to find a way to switch the horizontal and vertical lines such that we have the same number of intersections.

Combinatorial Problem 2 – Counting Dyck Paths with Square Paths

There is a known consequence of the Shuffle Conjecture that,

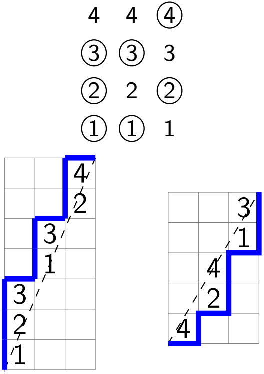
$$\sum_{\Pi \in \mathcal{D}_{m,n}} q^{coarea(\Pi)+dinv(\Pi)} = \frac{s_n[m]_q}{[m]_q} = \frac{1}{[m]_q} \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q .$$

The LHS is a sum over Dyck paths, and in the RHS, $\begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q$ is a sum over lattice path tracking coarea then divided by $[m]_q$.

Problem: can we correspond each Dyck path with m lattice paths to prove the identity directly?

Combinatorial Problem 3 – A Map on Labeled Dyck Paths

We discovered from experiment that there is a map on labeled Dyck paths keeping area and dinv.



area(LD1)=area(LD2) proved.

Problem: prove dinv(LD1)=dinv(LD2).

Thank You!