# Patterns in words of ordered set partitions

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**Abstract**

An ordered set partition of $[n] = \{1, 2, \ldots, n\}$ is a partition with an ordering on the parts. let $\mathcal{OP}_{n,k}$ be the set of ordered set partitions of $[n]$ with $k$ blocks, Godbole, Goyt, Herdan and Pudwell [4] defined $\mathcal{OP}_{n,k}(\sigma)$ to be the set of ordered set partitions in $\mathcal{OP}_{n,k}$ avoiding a permutation pattern $\sigma$ and obtained the formula for $|\mathcal{OP}_{n,k}(\sigma)|$ when the pattern $\sigma$ is of length 2. Later, Chen, Dai and Zhou in [2] found a formula algebraically for $|\mathcal{OP}_{n,k}(\sigma)|$ when the pattern $\sigma$ is of length 3.

In this paper, we define a new pattern avoidance for the set $\mathcal{OP}_{n,k}$, called $\mathcal{WOP}_{n,k}(\sigma)$, which includes the questions proposed in [4]. We obtain formulas for $|\mathcal{WOP}_{n,k}(\sigma)|$ combinatorially for any $\sigma$ of length $\leq 3$. We also study 3 kinds of descent statistics that we defined on $\mathcal{OP}_{n,k}(\sigma)$ for $\sigma$ of length $\leq 3$.

**Keywords:** permutations, ordered set partitions, pattern avoidance, bijections, Dyck paths

## 1 Introduction

In [4], Godbole, Goyt, Herdan, and Pudwell begin the study of patterns in ordered set partitions. In particular, they studied the number of ordered set partitions which avoid certain types of permutations of length 2 and 3. A partition $\pi$ of $[n] = \{1, \ldots, n\}$ is a family of nonempty, pairwise disjoint subsets $B_1, B_2, \ldots, B_k$ of $[n]$ called parts such that $\bigcup_{i=1}^{k} B_i = [n]$. We let $\ell(\pi)$ denote the number of parts in $\pi$ and $|\pi| = n$ denote the size of $\pi$. We let $\min(B_i)$ denote the minimal element of $B_i$ and we use the convention that we order the parts so that $\min(B_1) < \cdots < \min(B_k)$. To simplify notation, we shall write $\pi$ as $B_1/\ldots/B_k$. Thus we would write $\pi = 134/268/57$ for the set partition $\pi$ of [8] with parts $B_1 = \{1, 3, 4\}$, $B_2 = \{2, 6, 8\}$, and $B_3 = \{5, 6\}$. An ordered set partition with underlying set partition $\pi$ is just a permutation of the parts of $\pi$, $\delta = B_{\sigma_1}/\ldots/B_{\sigma_k}$ for some permutation $\sigma$ in the symmetric group $S_k$ . For example, $\delta = 57/134/268$ is an ordered set partition of [8] with underlying set partition $\pi = 134/268/57$. Given an ordered set partition $\delta = B_{\sigma_1}/\ldots/B_{\sigma_k}$, we let the word of $\delta$, $w(\delta)$, be the word obtained from $\delta$ by removing all the slashes. For example, if $\delta = 57/134/268$, then $w(\delta) = 57134268$. We let $\mathcal{OP}_n$ denote the set of ordered set partitions of $[n]$ and $\mathcal{OP}_{n,k}$ denote the set of ordered set partitions of $[n]$ with $k$ parts. If $b_1, \ldots b_k$ are positive integers, then we let

1. $\mathcal{OP}_{[b_1,\ldots,b_k]}$ denote the set of ordered set partitions $B_1/\ldots/B_k$ of $b_1 + \cdots + b_k$ such that $|B_i| = b_i$ for $i = 1, \ldots, b_k$,
2. $\mathcal{OP}_{n,\{b_1,\ldots,b_k\}}$ denote the set of ordered set partitions $\pi \in \mathcal{OP}_n$ such that the size of any part in $\pi$ is an element of $\{b_1, \ldots, b_k\}$, and

3. $\mathcal{OP}_{\langle b_1^{\beta_1}\ldots b_k^{\beta_k}\rangle}$ denote the set of ordered set partitions $\pi$ of $\sum_{i=1}^k \beta_i b_i$ which has $\beta_i$ parts of size $b_i$ for $i = 1, \ldots, k$.

Note that

$$\bigcup_{n\geq 0} \mathcal{OP}_{n,\{b_1,\ldots,b_k\}} = \bigcup_{\beta_1\geq 0,\ldots,\beta_k\geq 0} \mathcal{OP}_{\langle b_1^{\beta_1}\ldots b_k^{\beta_k}\rangle}.$$

Clearly, $\mathcal{OP}_{[b_1,\ldots,b_k]} = \binom{n}{b_1,\ldots,b_k}$, if $b_1 + \cdots + b_k = n$.

Given a sequence of distinct positive integers $w = w_1 \ldots w_n$, we let $\text{red}(w)$ denote the permutation in $S_n$ obtained from $w$ by replacing the $i$-th smallest letter in $w$ by $i$. For example, $\text{red}(4592) = 2341$. Following [4], we say that permutation $\sigma = \sigma_1 \ldots \sigma_j$ **occurs** in an ordered set partition $\delta = B_1/\ldots/B_k$ if and only if there exists $1 \leq i_1 < \cdots < i_j \leq k$ and $b_{i_j} \in B_{i_j}$ such that $\text{red}(b_{i_1} \ldots b_{i_j}) = \sigma$ and $\delta$ **avoids** $\sigma$ if $\sigma$ does not occur in $\delta$. For example, if $\delta = 57/134/268$, then 213 occurs in $\delta$ since $\text{red}(518) = 213$, but $\delta$ avoids 123 because every element in the first part $\{5, 7\}$ of $\delta$ is bigger than every element in the second part $\{1, 3, 4\}$ of $\delta$. If $\alpha$ is a permutation in $S_j$, then we let $\mathcal{OP}_n(\alpha)$ denote the set of odered set partions of $[n]$ that avoid $\alpha$. We can then define $\mathcal{OP}_{n,k}(\alpha)$, $\mathcal{OP}_{[b_1,\ldots,b_k]}(\alpha)$, $\mathcal{OP}_{n,\{b_1,\ldots,b_k\}}(\alpha)$, and $\mathcal{OP}_{\langle b_1^{\beta_1}\ldots b_k^{\beta_k}\rangle}(\alpha)$ in a similar manner. We let

$$\begin{aligned} op_n(\alpha) &= |\mathcal{OP}_n(\alpha)|, \\ op_{n,k}(\alpha) &= |\mathcal{OP}_{n,k}(\alpha)|, \\ op_{[b_1,\ldots,b_k]}(\alpha) &= |\mathcal{OP}_{[b_1,\ldots,b_k]}(\alpha)|, \text{and} \\ op_{\langle b_1^{\beta_1}\ldots b_k^{\beta_k}\rangle}(\alpha) &= |\mathcal{OP}_{\langle b_1^{\beta_1}\ldots b_k^{\beta_k}\rangle}(\alpha)|. \end{aligned}$$

Godbole, Goyt, Herdan, and Pudwell [4] proved a number of interesting results about this quantities. For example, they showed that

$$op_{n,k}(\sigma) = op_{n,k}(123)$$

for all permutations $\sigma$ of length 3. They also proved that

$$op_{n,3}(123) = op_{n,3}(123) = \left(\frac{n^2}{8} + \frac{3n}{8} - 2\right) 2^n + 3$$

and

$$op_{n,n-1}(123) = \frac{3(n-1)^2\binom{2n-2}{n-1}}{n(n+1)}.$$

Later, Chen, Dai, and Zhou [2] proved that

$$1 + \sum_{n\geq 1} t^n \sum_{k=1}^{n} op_{n,k}(123)x^k = \frac{-x + 2xt - 2t + 2t^2x + 2t^2 + x\sqrt{1 - 4xt - 4t + 4t^2x + 4t^2}}{2t(x+1)^2(t-1)}. \quad (1)$$

The goal of this paper is to study an alternative notion of pattern avoidance in ordered set partitions. Given an ordered set partition $\delta = B_1/\ldots/B_k$ of $[n]$, let $w(\delta) = w_1 \ldots w_n$ denote the word of $\delta$. Then we say that a permutation $\alpha = \alpha_1 \ldots \alpha_j \in S_j$ **occurs in the word of** $\delta$ if there exists $1 \leq i_1 < \cdots < i_j \leq n$ such that $\text{red}(w_{i_1} \ldots w_{i_j}) = \alpha$. Thus $\alpha$ occurs in the word of $\delta$ if $\alpha$ classically occurs in $w(\delta)$. We say that an ordered set partition $\delta$ **word-avoids** $\alpha$ if $\alpha$ does not occur in the word of $\delta$. For example, if $\delta = 57/134/268$, we saw that $\delta$ avoids 123 in the sense of [4], but clearly 123 occurs in the word of $\delta$ since $\text{red}(134) = 123$. Then we let $\mathcal{WOP}_n(\alpha)$ denote the set of ordered set partitions which word-avoid $\alpha$. Similarly, we can define $\mathcal{WOP}_{n,k}(\alpha)$, $\mathcal{WOP}_{[b_1,\ldots,b_k]}(\alpha)$, and $\mathcal{WOP}_{\langle b_1^{\beta_1}\ldots b_k^{\beta_k}\rangle}(\alpha)$. Then we let

$$\begin{aligned}
\mathrm{wop}_n(\alpha) &= |\mathcal{WOP}_n(\alpha)|,\\
\mathrm{wop}_{n,k}(\alpha) &= |\mathcal{WOP}_{n,k}(\alpha)|,\\
\mathrm{wop}_{[b_1,\ldots,b_k]}(\alpha) &= |\mathcal{WOP}_{[b_1,\ldots,b_k]}(\alpha)|, \text{and}\\
\mathrm{wop}_{\langle b_1^{\alpha_1}\ldots b_k^{\alpha_k}\rangle}(\alpha) &= |\mathcal{WOP}_{\langle b_1^{\alpha_1}\ldots b_k^{\alpha_k}\rangle}(\alpha)|.
\end{aligned}$$

We will also study the corresponding generating functions

$$\begin{aligned}
\mathbb{WOP}_\alpha(t) &= 1+\sum_{n\geq 1}\mathrm{wop}_n(\alpha)\ t^n,\\
\mathbb{WOP}_\alpha(x,t) &= 1+\sum_{n\geq 1}t^n\sum_{k=1}^{n}\mathrm{wop}_{n,k}(\alpha)\ x^k, \text{and}\\
\mathbb{WOP}_{\alpha,\{b_1,\ldots,b_n\}}(x,t,q_1,q_2,\ldots q_n) &= \sum_{\beta_1\geq 0}\cdots\sum_{\beta_k\geq 0}\mathrm{wop}_{\langle b_1^{\beta_1}\ldots b_k^{\beta_k}\rangle}(\alpha)\ t^{\sum_{i=1}^k b_i\beta_i}x^{\sum_{i=1}^k\beta_i}\ q_1^{\beta_1}q_2^{\beta_2}\cdots q_k^{\beta_k}.
\end{aligned}$$

Note that $\mathrm{wop}_{n,k}(321) = op_{n,k}(321)$. That is, if 321 occurs in the word of an ordered set partition $\delta$, then the occurrences of 3, 2, and 1 must have been in different parts of the partition $\delta$ so that 321 would occur in $\delta$ in the sense of Godbole, Goyt, Herdan, and Pudwell. However, for other $\sigma \in S_3$, it is not the case that $\mathrm{wop}_{n,k}(\sigma) = op_{n,k}(\sigma)$. In fact, it will follow from the results of the this paper that we have the following table of $\mathrm{wop}_n(\sigma)$ for $\sigma \in S_3$,

| $n$ | $\mathrm{wop}_n(123)$ | $\mathrm{wop}_n(132) = \mathrm{wop}_n(231) = \mathrm{wop}_n(312) = \mathrm{wop}_n(213)$ | $\mathrm{wop}_n(321)$ |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 |
| 2 | 3 | 3 | 3 |
| 3 | 9 | 11 | 12 |
| 4 | 31 | 45 | 56 |
| 5 | 113 | 197 | 284 |
| 6 | 431 | 903 | 1516 |
| 7 | 1697 | 4279 | 8384 |
| 8 | 6487 | 20793 | 47600 |
| 9 | 28161 | 103049 | 275808 |
| 10 | 117631 | 518859 | 1624352 |

We shall also study refinements of these generating functions by descents. In fact, there are three different natural notions of descents in an ordered set partition $\pi = B_1/\ldots/B_k$. That is, we let $\mathrm{des}(\pi)$ be the number of descents in the word of $\pi$, $w(\pi) = w_1 \ldots w_n$. Thus $\mathrm{des}(\pi) = |\{i : w_i > w_{i+1}\}|$. Given two consecutive parts $B_i$ and $B_{i+1}$, we write $B_i >_p B_{i+1}$ if every element of $B_i$ is greater than every element in $B_{i+1}$ and we write $B_i >_{min} B_{i+1}$ if the minimal element of $B_i$ is greater than the minimal element of $B_{i+1}$. We shall call elements $i$ such that $B_i >_p B_{i+1}$ part-descents and elements $i$ where $B_i >_{min} B_{i+1}$ min-descents. It is easy to see that if $B_i >_p B_{i+1}$, then $B_i >_{min} B_{i+1}$. However, if $\pi = 2/13$, then $2 >_{min} 13$ but it is not the case that $2 >_p 13$. We let $b_{i_{min}}$ be the minimum of block $B_i$ and $b_{i_{max}}$ be the maximum of block $B_i$, Then we define

$$\begin{aligned}
\mathrm{des}(\pi) &= |\{i : w(\pi)_i > w(\pi)_{i+1}\}| = |\{i : b_{i_{max}} > b_{i+1_{min}}\}|, \\
\mathrm{pdes}(\pi) &= |\{i : B_i >_p B_{i+1}\}| = |\{i : b_{i_{min}} > b_{i+1_{max}}\}| \quad \text{and} \\
\mathrm{mindes}(\pi) &= |\{i : B_i >_{min} B_{i+1}\}| = |\{i : b_{i_{min}} > b_{i+1_{min}}\}|.
\end{aligned}$$

These three statistics are not equi-distributed on ordered set partitions as the following table for ordered set partitions of 3 shows.

| $\pi$ | $\mathrm{des}(\pi)$ | $\mathrm{pdes}(\pi)$ | $\mathrm{mindes}(\pi)$ |
|---|---|---|---|
| 123 | 0 | 0 | 0 |
| 1/23 | 0 | 0 | 0 |
| 12/3 | 0 | 0 | 0 |
| 1/2/3 | 0 | 0 | 0 |
| 13/2 | 1 | 0 | 0 |
| 1/3/2 | 1 | 1 | 1 |
| 2/13 | 1 | 0 | 1 |
| 2/1/3 | 1 | 1 | 1 |
| 23/1 | 1 | 1 | 1 |
| 2/3/1 | 1 | 1 | 1 |
| 3/12 | 1 | 1 | 1 |
| 3/1/2 | 1 | 1 | 1 |
| 3/2/1 | 2 | 2 | 2 |

For each type of generating function above, we consider the refined generating function where we keep track of the number of descents of each type. For example, we shall study the following generating functions,

$$\begin{aligned}
\mathbb{WOP}_{\alpha}^{\mathrm{des}}(x,y,t) &= 1 + \sum_{n\geq 1} t^n \sum_{\pi\in\mathcal{OP}_n(\alpha)} x^{\ell(\pi)} y^{\mathrm{des}(\pi)}, \\
\mathbb{WOP}_{\alpha}^{\mathrm{pdes}}(x,y,t) &= 1 + \sum_{n\geq 1} t^n \sum_{\pi\in\mathcal{OP}_n(\alpha)} x^{\ell(\pi)} y^{\mathrm{pdes}(\pi)}, \text{ and} \\
\mathbb{WOP}_{\alpha}^{\mathrm{mindes}}(x,y,t) &= 1 + \sum_{n\geq 1} t^n \sum_{\pi\in\mathcal{OP}_n(\alpha)} x^{\ell(\pi)} y^{\mathrm{mindes}(\pi)}.
\end{aligned}$$

Similarly, we shall study

$$\begin{aligned}
\mathbb{WOP}_{\alpha,\{b_1,\ldots,b_k\}}^{\mathrm{des}}(x,y,t,q_1,\ldots,q_n) &= \sum_{\beta_1\geq 0,\ldots,\beta_k\geq 0,} \sum_{\pi\in\mathcal{WOP}_{\langle b_1^{\beta_1}\ldots b_k^{\beta_k}\rangle}(\alpha)} t^{|\pi|} x^{\ell(\pi)} y^{\mathrm{des}(\pi)} q_1^{\beta_1}\cdots q_k^{\beta_k}, \\
\mathbb{WOP}_{\alpha,\{b_1,\ldots,b_k\}}^{\mathrm{pdes}}(x,y,t,q_1,\ldots,q_n) &= \sum_{\beta_1\geq 0,\ldots,\beta_k\geq 0,} \sum_{\pi\in\mathcal{WOP}_{\langle b_1^{\beta_1}\ldots b_k^{\beta_k}\rangle}(\alpha)} t^{|\pi|} x^{\ell(\pi)} y^{\mathrm{pdes}(\pi)} q_1^{\beta_1}\cdots q_k^{\beta_k}, \text{ and} \\
\mathbb{WOP}_{\alpha,\{b_1,\ldots,b_k\}}^{\mathrm{mindes}}(x,y,t,q_1,\ldots,q_n) &= \sum_{\beta_1\geq 0,\ldots,\beta_k\geq 0,} \sum_{\pi\in\mathcal{WOP}_{\langle b_1^{\beta_1}\ldots b_k^{\beta_k}\rangle}(\alpha)} t^{|\pi|} x^{\ell(\pi)} y^{\mathrm{mindes}(\pi)} q_1^{\beta_1}\cdots q_k^{\beta_k}.
\end{aligned}$$

The main focus of this paper is the study the generating functions described above where $\alpha$ is $S_2$ or $S_3$. One advantage of our notion of word-avoidance in ordered set partitions is that we can employ standard techniques from the theory of generating functions and the Lagrange Inversion Theorem to give us nice answers. For example, we shall show that

$$\mathbb{WOP}_{132}(x,t) = \frac{t+1-\sqrt{(t+1)^2-4t(x+1)}}{2t(1+x)}, \tag{2}$$

$$\mathrm{wop}_{n,k}(132) = \frac{1}{k}\binom{n-1}{k-1}\binom{n+k}{k-1},$$

and

$$\mathrm{wop}_{\langle b_1^{\beta_1}\ldots b_k^{\beta_k}\rangle}(132) = \frac{1}{n}\binom{k}{\beta_1,\ldots,\beta_k}\binom{n+k}{n-1}$$

where $n=\sum_{i=1}^{k} b_i\beta_i$ and $k=\sum_{i=1}^{k}\beta_i$.

Similarly, we shall show that

$$\mathbb{WOP}_{132}^{\mathrm{des}}(x,y,t) = \frac{(1+2yt+xyt-t-xt)-\sqrt{((1+2yt+xyt-t-xt))^2-4t(1-t+ty)(x+yx)}}{2t(y+xy)}$$

and that

$$\mathrm{wop}_{n,k}(y,132) = \frac{1}{k}\binom{n-1}{k-1}\sum_{j-0}^{k-1}\binom{k}{j}\binom{n-1}{k-1-j}y^{k-1-j}$$

where $\mathrm{wop}_{n,k}(y,132) = \sum_{\pi\in\mathcal{OP}_{n,k}(132)} y^{\mathrm{des}(w(\pi))}$.

## 2 Preliminaries

The structure of elements in $\mathcal{WOP}_n(12)$ and $\mathcal{WOP}(21)$ are quite easy to describe. For example, if $\pi\in\mathcal{WOP}_n(12)$, then the word of $\pi$ must be $n(n-1)\ldots 21$ and hence $\pi = n/n-1/\ldots/1$. Similarly, if $\pi\in\mathcal{WOP}_n(21)$, then the word of $\pi$ must be $12\ldots(n-1)n$ and hence $\pi$ must be of the form $B_1/B_2/\ldots/B_k$ where for each $i=1,\ldots,k-1$, all the elements of $B_i$ are smaller than all the elements of $B_{i+1}$. It follows that $op_{n,k}(21)=\binom{n-1}{k-1}$ because to specify an ordered set partition $\pi\in\mathcal{OP}_{n,k}(21)$ with $k$ parts, we need only specify where we need to place $k-1$ /'s in the $n-1$ spaces between the letters of $1\ldots n$.

Thus

$$\mathbb{WOP}_{12}^{\mathrm{des}}(x,y,t) = 1+\sum_{n\geq 1} y^{n-1}x^n t^n = 1+\frac{xt}{1-xyt} \tag{3}$$

and $\mathbb{WOP}_{12}^{\mathrm{des}}(x,y,t) = \mathbb{WOP}_{12}^{\mathrm{pdes}}(x,y,t) = \mathbb{WOP}_{12}^{\mathrm{mindes}}(x,y,t)$. Also

$$\begin{aligned}
\mathbb{WOP}_{21}^{\mathrm{des}}(x,y,t) &= 1+\sum_{n\geq 1} t^n \sum_{k=1}^{n}\binom{n-1}{k-1}x^k \\
&= 1+xt\sum_{n\geq 1} t^{n-1}\sum_{k=1}^{n}\binom{n-1}{k-1}x^{k-1} \\
&= 1+xt\sum_{n\geq 1} t^{n-1}(1+x)^{n-1} \\
&= 1+\frac{xt}{1-t(1+x)}. \qquad (4)
\end{aligned}$$

and $\mathbb{WOP}_{21}^{\mathrm{des}}(x,y,t) = \mathbb{WOP}_{21}^{\mathrm{pdes}}(x,y,t) = \mathbb{WOP}_{21}^{\mathrm{mindes}}(x,y,t)$.

Next consider the generating functions $\mathbb{WOP}_{\alpha}^{\mathrm{des}}(x,y,t)$, $\mathbb{WOP}_{\alpha}^{\mathrm{pdes}}(x,y,t)$, and $\mathbb{WOP}_{\alpha}^{\mathrm{mindes}}(x,y,t)$ when $\alpha \in S_j$ for $j \geq 3$. There are some obvious symmetries in our situation. Recall that for a permutation $\sigma = \sigma_1 \ldots \sigma_n$, the reverse of $\sigma$, $\sigma^r$, is defined by $\sigma^r = \sigma_n \ldots \sigma_1$ and the complement of $\sigma$, $\sigma^c$, is defined by $\sigma^c = (n+1-\sigma_1)\ldots(n+1-\sigma_n)$. It is easy to see that $\mathrm{des}(\sigma) = \mathrm{des}((\sigma^r)^c)$. We can define reverse and complement on ordered set partitions as well. That is, suppose that $\pi = B_1/\ldots/B_k$ is an ordered set partition of $[n]$. Then if $B_i = \{a_1^i < a_2^i < \cdots < a_j^i\}$, we let complement of $B_i$, $B_i^c = \{(n+1-a_j^i) < \cdots < (n+1-a_2^i) < (n+1-a_1^i)\}$. Then we define the reverse of $\pi$, $\pi^r$, to be $B_k/\ldots/B_1$ and the complement of $\pi$, $\pi^c$, to be $B_1^c/\ldots/B_k^c$. Thus $(\pi^r)^c = B_k^c/\ldots/B_1^c$. It is easy to see that if $w(\pi) = w_1 \ldots w_n$, then the word of $(\pi^r)^c$ is $(n+1-w_n)\ldots(n+1-w_1) = (w(\pi)^r)^c$ Similarly it is easy to see that if $B_i >_p B_{i+1}$, then $B_{i+1}^c >_p B_i^c$. Thus the operations of reverse-complement show that

$$\begin{aligned}
\sum_{\pi \in \mathcal{OP}_{n,k}(\alpha)} x^{\ell(\pi)} y^{\mathrm{des}(\pi)} &= \sum_{\pi \in \mathcal{OP}_{n,k}((\alpha^r)^c)} x^{\ell(\pi)} y^{\mathrm{des}(\pi)} \text{ and} \\
\sum_{\pi \in \mathcal{OP}_{n,k}(\alpha)} x^{\ell(\pi)} y^{\mathrm{pdes}(\pi)} &= \sum_{\pi \in \mathcal{OP}_{n,k}((\alpha^r)^c)} x^{\ell(\pi)} y^{\mathrm{pdes}(\pi)}.
\end{aligned}$$

It follows that for all $1 \leq b_1 < \cdots < b_s$,

$$\begin{aligned}
\mathbb{WOP}_{132}^{*}(x,y,t) &= \mathbb{WOP}_{213}^{*}(x,y,t), \\
\mathbb{WOP}_{231}^{*}(x,y,t) &= \mathbb{WOP}_{312}^{*}(x,y,t), \\
\mathbb{WOP}_{132,\{b_1,\ldots,b_s\}}^{*}(x,y,t,q_1,\ldots,q_s) &= \mathbb{WOP}_{213,\{b_1,\ldots,b_s\}}^{*}(x,y,t,q_1,\ldots,q_s), \text{ and} \\
\mathbb{WOP}_{231,\{b_1,\ldots,b_s\}}^{*}(x,y,t,q_1,\ldots,q_s) &= \mathbb{WOP}_{312,\{b_1,\ldots,b_s\}}^{*}(x,y,t,q_1,\ldots,q_s).
\end{aligned}$$

where $*$ is either des or pdes.

Reverse-complement does not always preserve mindes. For example, one can easily see from the table of values of $\mathrm{mindes}(\pi)$ for $\pi \in \mathcal{OP}_3$ that, it is not the case that

$$\sum_{\pi \in \mathcal{OP}_3(132)} x^{\ell(\pi)} y^{\mathrm{mindes}(\pi)} = \sum_{\pi \in \mathcal{OP}_3(213)} x^{\ell(\pi)} y^{\mathrm{mindes}(\pi)}.$$

In general, reverse and complement by themselves do not preserve these generating functions. For example, it follows from the same table that

$$\begin{aligned}
\sum_{\pi \in \mathcal{OP}_n(123)} x^{\ell(\pi)} y^{\mathrm{des}(\pi)} &\neq \sum_{\pi \in \mathcal{OP}_n(321)} x^{\ell(\pi)} y^{\mathrm{des}(\pi)}, \\
\sum_{\pi \in \mathcal{OP}_n(123)} x^{\ell(\pi)} y^{\mathrm{pdes}(\pi)} &\neq \sum_{\pi \in \mathcal{OP}_n(321)} x^{\ell(\pi)} y^{\mathrm{pdes}(\pi)}, \text{ and} \\
\sum_{\pi \in \mathcal{OP}_n(123)} x^{\ell(\pi)} y^{\mathrm{mindes}(\pi)} &\neq \sum_{\pi \in \mathcal{OP}_n(321)} x^{\ell(\pi)} y^{\mathrm{mindes}(\pi)}.
\end{aligned}$$

Our next theorem will show that

$$\mathbb{WOP}_{312}^{\mathrm{des}}(x,y,t) = \mathbb{WOP}_{213}^{\mathrm{des}}(x,y,t)$$

and

$$\mathbb{WOP}_{312}^{\text{mindes}}(x,y,t) = \mathbb{WOP}_{213}^{\text{mindes}}(x,y,t).$$

Thus there are only three different generating functions of the form $\mathbb{WOP}_{\alpha}^{\text{des}}(x,y,t)$ for $\alpha \in S_3$. Similarly, our next theorem will show that for all $1 \le b_1 < \cdots < b_s$,

$$\mathbb{WOP}_{213,\{b_1,\ldots,b_s\}}^{\text{des}}(x,y,t,q_1,\ldots,q_s) = \mathbb{WOP}_{312,\{b_1,\ldots,b_s\}}^{\text{des}}(x,y,t,q_1,\ldots,q_s)$$

and

$$\mathbb{WOP}_{213,\{b_1,\ldots,b_s\}}^{\text{mindes}}(x,y,t,q_1,\ldots,q_s) = \mathbb{WOP}_{312,\{b_1,\ldots,b_s\}}^{\text{mindes}}(x,y,t,q_1,\ldots,q_s).$$

**Theorem 1.** *There is a bijection $\phi_n : \mathcal{WOP}_n(312) \to \mathcal{WOP}_n(213)$ such that for all $\pi = B_1/\ldots/B_k \in \mathcal{WOP}_n(213)$, $\phi_n(\pi) = C_1/\ldots/C_k \in \mathcal{WOP}_n(312)$ where $|B_i| = |C_i|$ for $i = 1,\ldots,k$, 1 in position $k$ in $w(\pi)$ if and only if 1 is in position $k$ in $w(\phi_n(\pi))$, $\text{des}(\pi) = \text{des}(\phi_n(\pi))$, $\text{Des}(w(\pi)) = \text{Des}(w(\phi_n(\pi)))$, and $\text{mindes}(\pi) = \text{mindes}(\phi_n(\pi))$.*

*Proof.* We shall define $\phi_n : \mathcal{WOP}_n(312) \to \mathcal{WOP}_n(213)$ by induction on $n$. For $1 \le n \le 2$, we let $\phi_n$ be the identity. Now assume that we have defined $\phi_k : \mathcal{WOP}_k(312) \to \mathcal{WOP}_k(213)$ for $k \le n-1$. We classify the ordered set partitions $\pi$ in $\mathcal{WOP}_n(312)$ by the position of 1 in $w(\pi)$. First suppose that 1 occurs in position 1 in $w(\pi)$. If 1 in a part by itself, then $\pi$ is of the form $1/B_2/\ldots/B_k$ for some $k \ge 2$. In this case, if we subtract 1 for each element in $B_2/\ldots/B_k$ to obtain a set partition $\pi^* = B_2^*/\ldots/B_k^*$ in $\mathcal{WOP}_{n-1}(312)$. Then let $\phi_{n-1}(B_2^*/\ldots/B_k^*) = C_2^*/\ldots/C_k^*$ and let $C_2/\ldots/C_k$ be result of adding 1 to each element of $C_2^*/\ldots/C_k^*$. It is easy to see that if we let $\phi_n(1/B_2/\ldots/B_k) = 1/C_2/\ldots/C_k$, then $1/C_2/\ldots/C_k \in \mathcal{WOP}_1(213)$, $|B_i| = |C_i|$ for $i = 2,\ldots,k$, $\text{des}(1/B_2/\ldots/B_k) = \text{des}(1/C_2/\ldots/C_k)$, $\text{Des}(w(1/B_2/\ldots/B_k)) = \text{Des}(w(1/C_2/\ldots/C_k))$, and $\text{mindes}(1/B_2/\ldots/B_k) = \text{mindes}(1/C_2/\ldots/C_k)$. If 1 is not in a part by itself, then $\pi$ is of the form $B_1/\ldots/B_k$ where $1 \in B_1$ and $|B_1| \ge 2$. In this case, we can remove 1 from $B_1$ and subtract 1 for each of the remaining elements to obtain an ordered set partition $\pi^* = B_1^*/\ldots/B_k^*$ in $\mathcal{WOP}_{n-1}(312)$. Then let $\phi_{n-1}(B_1^*/\ldots/B_k^*) = C_1^*/\ldots/C_k^*$ and let $C_1/\ldots/C_k$ be result of adding 1 to each element of $C_1^*/\ldots/C_k^*$ and then adding 1 to the first part. Again it is easy to see that if we let $\phi_n(B_1/\ldots/B_k) = C_1/\ldots/C_k$, then $C_1/\ldots/C_k \in \mathcal{WOP}_1(213)$, $|B_i| = |C_i|$ for $i = 1,\ldots,k$, $\text{des}(B_1/\ldots/B_k) = \text{des}(C_1/\ldots/C_k)$, $\text{Des}(w(B_1/\ldots/B_k)) = \text{Des}(w(C_1/\ldots/C_k))$, and $\text{mindes}(B_1/\ldots/B_k) = \text{mindes}(C_1/\ldots/C_k)$.

Next suppose that $\pi \in WOP_n(312)$ is such that 1 is position $r$ in $w(\pi)$ where $r \ge 2$. Then $\pi$ must be form $B_1/\ldots/B_j/B_{j+1}/\ldots/B_k$ where $j \ge 1$ and 1 is the first element in part $B_{j+1}$. Since $w(\pi)$ is 312-avoiding, it must be the case all the elements of $B_1/\ldots/B_j$ are less than all of the elements $B_{j+1} - \{1\}, B_{j+2}\ldots B_k$. It follows that $B_1/\ldots/B_j$ is a set partition of $\{2,\ldots,r\}$ such that $w(B_1/\ldots/B_j)$ reduces to a 312-avoiding permutation and $B_{j+1} - \{1\}/\ldots/B_k$ is a set partition of $\{r+1,\ldots,n\}$ such that reduction of $w(B_{j+1}\ldots B_k)$ is 312-avoiding. Moreover, $r-1$ a descent in $w(\pi)$ and $B_j >_{min} B_{j+1}$. In this case, we let $B_{j+1}^*/\ldots/B_k^*$ be the result of subtracting $r-1$ from each element of $B_{j+1}\ldots B_k$ except 1 so that $B_{j+1}^*/\ldots/B_k^*$ is an ordered set partition of $\mathcal{WOP}_{n-r+1}(312)$ whose word starts with 1. We let $B_1^*/\ldots/B_j^*$ be the result of subtracting 1 from each element of $B_1/\ldots/B_j$ so that $B_1^*/\ldots/B_j^*$ is an element of $\mathcal{WOP}_{k-1}(312)$.

Now let $\phi_{k-1}(B_1^*/\ldots/B_j^*) = C_1/\ldots/C_k$ and $\phi_{n-k-1}(B_{j+1}^*/\ldots/B_k^*) = D_1/\ldots/D_{k-j}$. We can then add $n-r$ to each element of $C_1/\ldots/C_j$ to produce an ordered set partition $C_1^*/\ldots/C_j^*$ of $\{n-r+1,\ldots,n\}$ whose word reduces to 213-avoiding permutation such that $\text{des}(\text{red}(w(C_1^*/\ldots/C_j^*))) = \text{des}(w(B_1/\ldots/B_j))$, $\text{Des}(\text{red}(w(C_1^*/\ldots/C_j^*))) = \text{Des}(w(B_1/\ldots/B_j))$, and $\text{mindes}(C_1^*/\ldots/C_j^*) = \text{mindes}(B_1/\ldots/B_j)$. Then we let

$$\phi_n(\pi) = C_1^*/\ldots/C_j^*/D_1/\ldots/D_{k-j}.$$

It is easy to see by induction that $\mathrm{des}(w(\pi)) = \mathrm{des}(w(\phi_n(\pi)))$, $\mathrm{Des}(w(\pi)) = \mathrm{Des}(w(\phi_n(\pi)))$ and $\mathrm{mindes}(\pi)) = \mathrm{mindes}(\phi_n(\pi))$. Moreover, by construction 1 is in position $r$ in both $w(\pi)$ and $w(\phi_n(\pi))$. The only thing that we have to check is the $w(\phi_n(\pi))$ is 213-avoiding. But this follows from the fact that all the elements in $C_1^*/\dots/C_j^*$ are bigger than all the elements in $D_1/\dots/D_{k-j}$ and the word of $C_1^*/\dots/C_j^*$ reduces to a 213-avoiding permutation and the word of $D_1/\dots/D_{k-j}$ is a 231-avoiding permutation. □

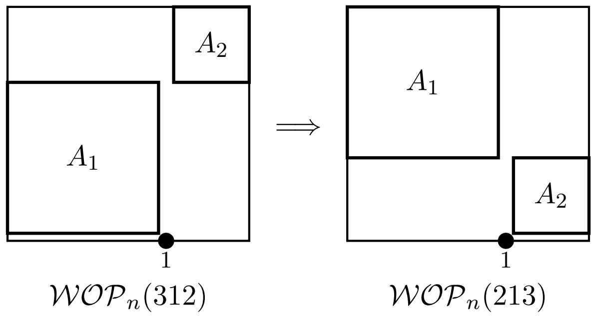


Figure 1: Bijection $\phi_n : \mathcal{WOP}_n(312) \to \mathcal{WOP}_n(213)$

Following is an example that $\phi_5(3/24/15) = 5/34/12$ keeps descents, Descent set of word and min-descents.

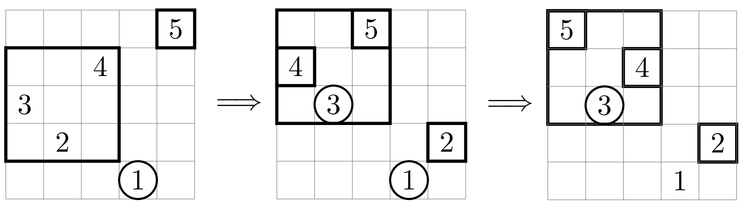

Figure 2: $\pi = 3/24/15 \in \mathcal{WOP}_{5,3}(312) \Rightarrow \phi_5(\pi) = 5/34/12 \in \mathcal{WOP}_{5,3}(213)$

We end this section with two observations. Suppose that $\pi = B_1/\dots/B_k \in \mathcal{WOP}_{n,k}(132)$. First, we notice that if the last element $\ell_i$ of $B_i$ is greater than the first element of $B_{i+1}$ so that there is a descent in $w(\pi)$ at position $\sum_{j=1}^{i} |B_j|$, then it must be the case that $\min(B_i) > \min(B_{i+1})$. That is, if $\min(B_i) < \min(B_{i+1})$, then $\min(B_i) \neq \ell_i$ and hence $(b_{i_{min}}, \ell_i, b_{i+1_{min}})$ would reduce to 132. It follows that for all $\pi \in \mathcal{WOP}_n(132)$, $\mathrm{des}(\pi) = \mathrm{mindes}(\pi)$ and, hence,

$$\mathbb{WOP}_{132}^{\mathrm{des}}(x, y, t) = \mathbb{WOP}_{132}^{\mathrm{mindes}}(x, y, t).$$

Second, if we let $i$ such that $\max(B_i) > \max(B_{i+1})$ be a max-descent and let $\mathrm{maxdes}(\pi)$ be the number of max-descent of an ordered set partition $\pi$, then for any $\pi = B_1/\dots/B_k \in \mathcal{WOP}_{n,k}(132)$, $i$ is a max-descent if and only if $i$ is a part-descent. Otherwise if $b_{i_{min}} < b_{i+1_{max}}$, then the triple $(b_{i_{min}}, b_{i_{max}}, b_{i+1_{max}})$ matches pattern 132. Thus,

$$\mathbb{WOP}_{132}^{\mathrm{pdes}}(x, y, t) = \mathbb{WOP}_{132}^{\mathrm{maxdes}}(x, y, t).$$

## 3 Computing $\mathbb{WOP}^{\mathrm{des}}_{\alpha}(x,y,t)$ for $\alpha \in S_3$

In this section, we shall derive the generating functions $\mathbb{WOP}^{\mathrm{des}}_{\alpha}(x,y,t)$ for $\alpha \in S_3$. We start by considering $\mathbb{WOP}^{\mathrm{des}}_{132}(x,y,t)$. In this case, we shall classify the ordered set partitions $\pi$ in $\mathcal{WOP}_n(132)$ by the size of last part. That is, suppose that $\pi = B_1/\ldots/B_k$ where $B_k = \{a_1 < \cdots < a_r\}$. Then we let $A_{r+1}$ denote the set of elements in $B_1/\ldots/B_{k-1}$ that are greater that $a_r$, $A_1$ denote the set of elements in $B_1/\ldots/B_{k-1}$ that are less that $a_1$, and $A_i$ denote the set of elements $j$ in $B_1/\ldots/B_{k-1}$ such that $a_i > j > a_{i-1}$ for $i = 2,\ldots,r$. Since $w(\pi)$ is 132-avoiding, for any $i \geq 2$, every element $y$ in $A_i$ must appear to the left of every element $x$ in $A_{i-1}$ since otherwise $xya_i$ would be an occurrence of 132 in $w(\pi)$. It follows that the word of $\pi$ has the structure pictured in Figure 3. Note that it is possible that any given $A_i$ is empty. However, this structure ensures that no part of $\pi$ can contain elements two different $A_i$'s so that if $A_i$ is non-empty, then $A_i$ is a union of consecutive parts of $\pi$, say $A_i = B_1^{(i)}/\ldots/B_{j_i}^{(i)}$. Moreover, if $i \geq 2$ and $A_i \neq \emptyset$, then the last element of $B_{j_i}^{(i)}$ is a descent in $w(\pi)$. That is, either $A_1,\ldots,A_{i-1}$ are empty and there is a descent from the last element of $B_{j_i}^{(i)}$ to $a_1$ which is the first element of $B_k$ or one of one of $A_1,\ldots,A_{i-1}$ is non-empty. Then we let $p$ be the largest integer $r$ such that $1 \leq r \leq i-1$ and $A_r$ is non-empty. Then there is a descent from the last element of $B_{j_i}^{(i)}$ to the first element of the first part of $A_p$.

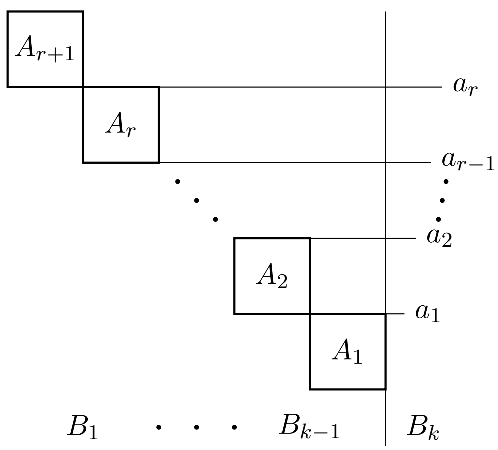


Figure 3: The structure of $\pi \in \mathcal{WOP}_n(132)$.

Now suppose that $B(x,y,t) = \mathbb{WOP}^{\mathrm{des}}_{132}(x,y,t)$. Then this structure implies that $B(x,y,t)$ satisfies the following recursive relation.

$$B(x,y,t) = 1 + \sum_{r\geq 1} xt^r(1+y(B(x,y,t)-1))^r B(x,y,t). \tag{5}$$

In (5) the factor $xt^r$ accounts for those ordered set partitions $\pi$ whose last part is of size $r$. We get a factor $1+y(B(x,y,t)-1)$ for $A_i$ for $i = 2,\ldots,r+1$ where the 1 accounts for the possibility that $A_i$ is empty and the term $y(B(x,y,t)-1)$ accounts for the fact that there is descent starting at the last element of $A_i$ if $A_i$ is non-empty. Finally the last factor $B(x,y,t)$ corresponds the contribution over all possible $A_1$.

It follows that

$$B(x,y,t) = 1 + \frac{xtB(x,y,t)(1+y(B(x,y,t)-1))}{1-t(1+y(B(x,y,t)-1))}. \tag{6}$$

Multiplying both sides of (6) by $1-t(y(B(x,y,t)-1))$ leads to the quadratic equation,

$$0=(1-t+ty)-B(x,y,t)(1-2yt+xyt-t-tx)+t(xy+y)(B(x,y,t)^2 \tag{7}$$

and solving for $B(x,y,t)$ gives that

$$B(x,y,t)=\frac{(1+2yt+xyt-t-tx)-\sqrt{(1+2yt+xyt-t-tx)^2-4(1-t+ty)(t(y+xy))}}{2t(y+yt)}. \tag{8}$$

If we let $f(x,y,t)=B(x,y,t)-1$, then (6) gives that

$$f(x,y,t)=x\frac{t(f(x,y,t)+1)(1+y(f(x,y,t))}{1-t(1+yf(x,y,t))}. \tag{9}$$

The Lagrange Inversion Theorem implies that the coefficient of $x^k$ in $f(x,y,t)$, $f(x,y,t)|_{x^k}$, is given by

$$f(x,y,t)|_{x^k}=\frac{1}{k}\Gamma(x)^k|_{x^{k-1}}$$

where $\Gamma(x)=\frac{t(x+1)(1+yx)}{1-t(1+yx)}$. Using Newton's binomial theorem, we can compute that

$$\begin{aligned}
f(x,y,t)|_{x^kt^n} &= \frac{1}{k}\frac{t^k(1+x)^k(1+yx)^k}{(1-t(1+yx))^k}|_{x^{k-1}t^n}\\
&= \frac{1}{k}(1+x)^k(1+yx)^k\left(\sum_{s\geq 0}\binom{k+s-1}{k-1}t^s(1+xy)^s\right)|_{x^{k-1}t^{n-k}}\\
&= \frac{1}{k}(1+x)^k(1+yx)^n\binom{k+n-k-1}{k-1}|_{x^{k-1}}\\
&= \frac{1}{k}\binom{n-1}{k-1}\sum_{j=0}^{k-1}\binom{k}{j}\binom{n}{k-1-j}y^{k-1-j}.
\end{aligned}$$

Thus we have the following theorem.
**Theorem 2.**

$$\mathbb{WOP}_{132}^{\mathrm{des}}(x,y,t)=$$
$$\frac{(1+2yt+xyt-t-tx)-\sqrt{(1+2yt+xyt-t-tx)^2-4(1-t+ty)(t(y+xy))}}{2t(y+yx)} \tag{10}$$

*and*

$$\sum_{\pi\in\mathcal{WOP}_{n,k}(132)}y^{\mathrm{des}(\pi)}=\frac{1}{k}\binom{n-1}{k-1}\sum_{j=0}^{k-1}\binom{k}{j}\binom{n}{k-1-j}y^{k-1-j}. \tag{11}$$

Below are the first few terms of $\mathbb{WOP}_{132}^{\mathrm{des}}(x,y,t)$.

$$\begin{aligned}
&\mathbb{WOP}_{132}^{\mathrm{des}}(x,y,t)=\\
&1+xt+(x+(1+y)x^2)t^2+(x+(2+3y)x^2+(1+3y+y^2))t^3+\\
&(x+(3+6y)x^2+(3+12y+6y^2)x^3+(1+6y+6y^2+y^3)x^4)t^4+\\
&\big(x+(4+10y)x^2+(6+30y+20y^2)x^3+(4+30y+40y^2+10y^3)x^4+\\
&(1+10y+20y^2+10y^3+y^4)x^5\big)\,t^5+\cdots
\end{aligned}$$

Setting $y = 1$ in Theorem 2 and observing that $\sum_{j=0}^{k-1} \binom{n}{j}\binom{n}{k-1-j} = \binom{n+k}{k-1}$, we have the following corollary.

**Corollary 3.**

$$\mathbb{WOP}_{132}^{\mathrm{des}}(x,1,t) = \frac{(1+t) - \sqrt{(1+t)^2 - 4t(1+x)}}{2t(1+x)} \tag{12}$$

*and*

$$wop_{n,k}(132) = \frac{1}{k}\binom{n-1}{k-1}\binom{n+k}{k-1}. \tag{13}$$

Below are the first few terms of $\mathbb{WOP}_{132}^{\mathrm{des}}(x,1,t)$.

$$\begin{aligned}
&\mathbb{WOP}_{132}^{\mathrm{des}}(x,1,t) = \\
&1 + xt + (x+2x^2)t^2 + (x+5x^2+5x^3)t^3 + \\
&(x+9x^2+21x^3+14x^4)t^4 + (x+14x^2+56x^3+84x^4+42x^5)t^5 + \\
&(x+20x+120x^3+300x^4+330x^5+132x^6)t^6 + \\
&(x+27x+225x^3+825x^4+1485x^5+1287x^6+429x^7)t^7 + \\
&(x+35x+385x^3+1925x^4+5005x^5+7007x^6+5005x^7+1430x^8)t^8 + \\
&(x+44x+616x^3+4004x^4+14014x^5+28028x^6+32032x^7+19448x^8+4862x^9)t^9 + \cdots
\end{aligned}$$

It follows from Theorem 2 that $\mathrm{wop}_n(132)$ is the number of rooted planar trees with $n+1$ leaves that have no vertices of out degree 1. The reason is that the generating function $F$ of both objects when only track object size $n$ satisfies the recursion that

$$F(t) = 1 + \sum_{r \geq 1} t^r F(t)^{r+1}.$$

A bijection follows naturally from the generating function. Figure 4 shows an example of the bijection. Based on the recursion, the number of non-leaves is equal to the number of blocks of the ordered set partition, and the number of out degree of the root is equal to (the size of the last block)+1 of the ordered set partition.

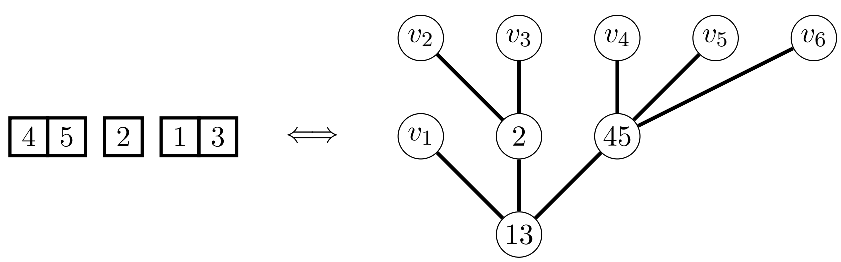


Figure 4: Bijection between $\mathcal{WOP}_n(132)$ and rooted planar trees with no vertices out degree 1

Given any sequence of positive numbers $1 \leq b_1 < b_2 < \cdots < b_s$, we let

$$A = A(x,y,t,q_1,\ldots,q_s) = \mathbb{WOP}_{132,\{b_1,\ldots,b_s\}}^{\mathrm{des}}(x,y,t,q_1,\ldots,q_s).$$

It follows from the block structure pictured in Figure 3 that

$$A = 1 + \sum_{i=1}^{s} xq_i t^{b_i}(1+y(A-1))^{b_i} A.$$

If we set $F = F(x,y,t,q_1,\ldots,q_s) = A(x,y,t,q_1,\ldots,q_s) - 1$, then we see that

$$F = x(F+1)\sum_{i=1}^{s} q_i t^{b_i}(1+yF)^{b_i}.$$

It follows from Lagrange Inversion that

$$F|_{x^k} = \frac{1}{k}\delta^k(x)|_{x^{k-1}}$$

where $\delta(x) = (x+1)\sum_{i=1}^{s} q_i t^{b_i}(1+yx)^{b_i}$. Thus

$$\begin{aligned}
F|_{x^k t^n} &= \frac{1}{k}(x+1)^k \sum_{\substack{\alpha_i \geq 0\\ \alpha_1+\cdots+\alpha_s=k}} \binom{k}{\alpha_1,\ldots,\alpha_s} t^{\sum_{i=1}^{k}\alpha_i b_i}(1+yx)^{(\sum_{i=1}^{k}\alpha_i b_i)} \prod_{i=1}^{s} q_i^{\alpha_i}|_{x^{k-1}t^n} \\
&= \frac{1}{k}(x+1)^k(1+yx)^n \sum_{\substack{\alpha_1+\cdots+\alpha_s=k\\ \alpha_1 b_1+\cdots\alpha_k b_k=n}} \binom{k}{\alpha_1,\ldots,\alpha_s}\prod_{i=1}^{s} q_i^{\alpha_i}|_{x^{k-1}} \\
&= \frac{1}{k}\left(\sum_{j=0}^{k-1}\binom{k}{j}\binom{n}{k-1-j}y^{k-1-j}\right) \sum_{\substack{\alpha_1+\cdots+\alpha_s=k\\ \alpha_1 b_1+\cdots\alpha_k b_k=n}} \binom{k}{\alpha_1,\ldots,\alpha_s}\prod_{i=1}^{s} q_i^{\alpha_i}.
\end{aligned}$$

If $\sum \alpha_i b_i = n$, then taking the coefficient of $q_1^{\alpha_1}\ldots q_s^{\alpha_s}$ on both sides of the above expression yields the following theorem.

**Theorem 4.** *Suppose that* $0 < b_1 < \cdots < b_s$, $\sum_{i=1}^{s} = k$, *and* $\sum \alpha_i b_i = n$. *Then*

$$\sum_{\pi \in \mathcal{WOP}_{\langle b_1^{\alpha_1}\ldots b_s^{\alpha_s}\rangle}(132)} y^{\mathrm{des}(\pi)} = \frac{1}{k}\binom{k}{\alpha_1,\ldots,\alpha_s}\left(\sum_{j=0}^{k-1}\binom{k}{j}\binom{n}{k-1-j}y^{k-1-j}\right).$$

Setting $y = 1$ in Theorem 4 and observing that $\sum_{j=0}^{k-1}\binom{k}{j}\binom{n}{k-1-j} = \binom{n+k}{k-1}$ yields the following corollary.

**Corollary 5.** *Suppose that* $0 < b_1 < \cdots < b_s$, $\sum_{i=1}^{s}\alpha_i = k$, *and* $\sum_{i=1}^{s}\alpha_i b_i = n$. *Then*

$$\mathrm{wop}_{\langle b_1^{\alpha_1}\ldots b_s^{\alpha_k}\rangle}(132) = \frac{1}{k}\binom{n+k}{k-1}\binom{k}{\alpha_1,\ldots,\alpha_s}.$$

Next we turn our attention to ordered set partitions $\pi$ such that $w(\pi)$ avoids 123. In this case, all the parts of $\pi$ must be of size 1 or 2 since any part $B$ of size $k \geq 3$ immediately yields an consecutive increasing sequence of size $k$ in the word of its ordered set partition.

Thus we will compute the generating function

$$\mathbb{WOP}^{\mathrm{des}}_{123,\{1,2\}}(x,y,t,q_1,q_2) = \sum_{\pi \in \mathcal{WOP}\langle 1^k 2^\ell\rangle} y^{\mathrm{des}(\pi)} t^{k+2\ell} x^{k+\ell} q_1^k q_2^\ell.$$

To compute $\mathbb{WOP}_{123,\{1,2\}}(x,y,t,q_1,q_2)$, we must first review a bijection of Deutsch and Elizalde between 123-avoiding permutations and Dyck paths.

Given an $n \times n$ chessboard , we set the origin $(0,0)$ at the lower left corner, and label the coordinates of the columns from left to right with $0,1,\ldots,n$ and the coordinates of the rows from bottom to top with $0,1\ldots,n$. A Dyck path is a path made up of unit down-steps $D$ and unit right-steps $R$ which starts at $(0,n)$, which is at the bottom right-hand corner, and ends at $(n,0)$, which is at the top left-hand corner, has stays on or below the diagonal $x=y$. Given a Dyck path $P$, we let

$$Return(P) = \{i \geq 1 : P \text{ goes through the point } (i, n-i)\}$$

and we let $\text{return}(P) = |Return(P)|$. For example, for the Dyck path

$$P = DDRDDRRRDDRDRDRRDR$$

shown on the right in Figure 5, $Return(P) = \{4,8,9\}$ and $\text{return}(P) = 3$.

Given any permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n(123)$, we write it on our $n \times n$ chessboard by placing $\sigma_i$ in the $i^{\text{th}}$ column and $\sigma_i^{\text{th}}$ row, reading from bottom to top. Then, we shade the cells to the north-east of the cell that contains $\sigma_i$. The path $\Psi(\sigma)$ is the path that goes along the south-west boundary of the shaded cells. For example, this process is pictured in Figure 5 for the permutation $\sigma = 869743251 \in S_9(123)$ which maps to the Dyck path $DDRDDRRRDDRDRDRRDR$.

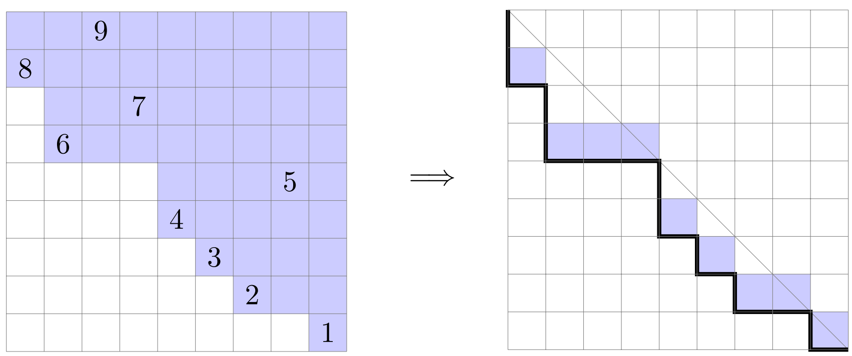


Figure 5: $S_n(123)$ to $\mathcal{D}_n$

Given any Dyck path $P$, we construct $\Psi^{-1}(P) = \sigma_{123}(P)$ as follows. First we place an "$\times$" in every outer corner of $P$. Then we consider the rows and columns which do not have a $\times$. Processing the columns from top to bottom and the rows from left to right, we place an $\times$ in the $i^{\text{th}}$ empty row and $i^{\text{th}}$ empty column. This process is pictured in Figure 6. The details that $\Psi$ is bijection between $S_n(123)$ and $\mathcal{D}_n$ can be found in [3].

We shall classify the ordered set partitions $\pi \in \mathcal{WOP}_n(123)$ by the first return (from left to right) of the path $\Psi(w(\pi)) = P$. Suppose that the first return of the path $P$ is at the point $(n-k,k)$, then the path $P$ is divided by the first return into 2 paths, path $DAR$ and path $B$, as shown in Figure 7 $(a)$. The numbers in the corners above the point $(n-k,k)$ must come from $\{k+1,\ldots n\}$. Because we place the $\times$s in the columns which are not occupied by the $\times$s in the outer corners of $P$, in a decreasing manner, reading from left to right, it follows that by the time we have reached column $n-k$, we must have used all the numbers $\{k+1,\ldots,n\}$. This means that there can be no $\times$s in the red area of the diagram so that all the $\times$s in the last $k$ columns must lie in lower $k$ rows. In particular, this implies in that in $w(\pi)$, all the elements in $\{k+1,\ldots,n\}$ preceed all the elements in $\{1,\ldots,k\}$. The elements in $\{k+1,\ldots,n\}$ are based on path $DAR$ and the elements in $\{1,\ldots,k\}$ are based on path $B$, and there is a descent between the last of occurrence of a letter in $\{k+1,\ldots,n\}$ in $w(\pi)$ and the first occurrence of a letter in $\{1,\ldots,k\}$ in $w(\pi)$ if $k>0$. Hence we

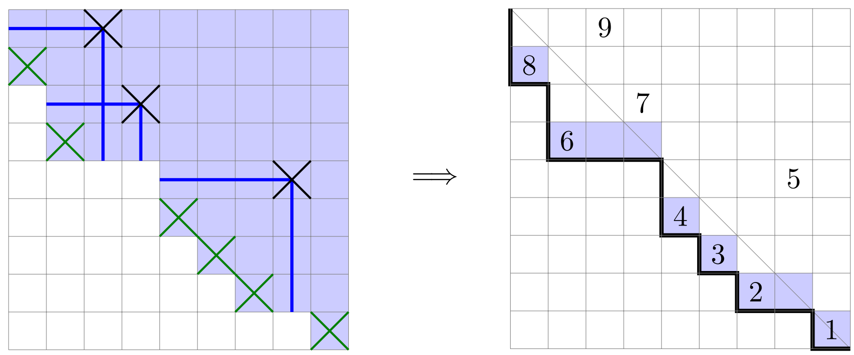


Figure 6: $\mathcal{D}_n$ to $S_n(123)$

must be able to break any ordered set partition $\pi = B_1/\dots/B_j$ such that $\Psi(w(\pi)) = P$ into two parts, $B_1/\dots/B_i$ which contains all the letters in $\{k+1,\dots,n\}$ and $B_{i+1}/\dots/B_j$ which contain all the letters in $\{1,\dots,k\}$. Let $A(x,y,t,q_1,q_2) = \mathbb{WOP}^{\text{des}}_{123,\{1,2\}}(x,y,t,q_1,q_2)$. It is easy to see that the contribution to $A(x,y,t,q_1,q_2)$ by summing over the weights of all possible choices of $B_{i+1}/\dots/B_j$ as $k$ varies over all choices of $k>0$ is $y(A(x,y,t,q_1,q_2)-1)$ and is equal to 1 if $k=0$.

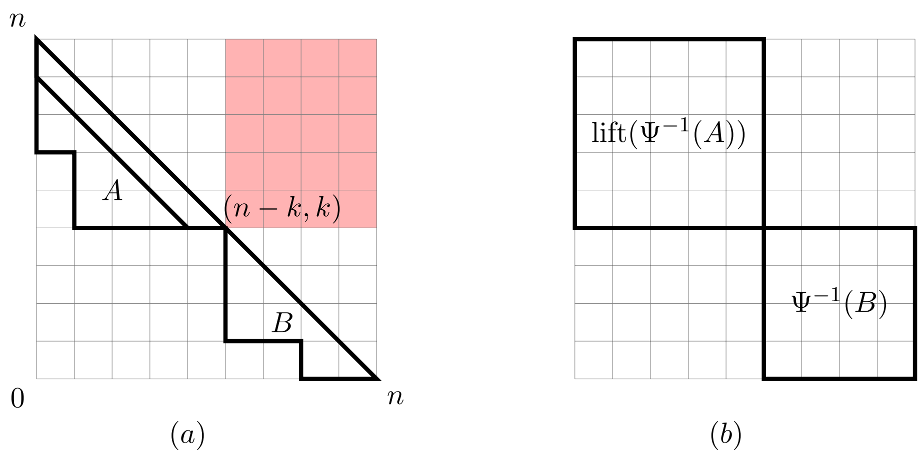


Figure 7: The first return of $P$

To analyze the contribution from parts $B_1/\dots/B_i$, we need to work on path $DAR$, which can be seen as lifting the path $A$ one unit higher. We let $\text{lift}(P)$ be the path $DPR$. For $\sigma \in S_n(123)$ and $P = \Psi^{-1}(\sigma)$, we write $\text{lift}(\sigma)$ for the permutation $\Psi^{-1}(\text{lift}(P)) = \Psi^{-1}(DPR) \in S_{n+1}$ corresponding with path $\text{lift}(P)$.

The lifting operation is pictured in Figure 8. We say that a pair of consecutive $DR$ steps is a peak of a Dyck path, and in the corresponding 123-avoiding permutation, the numbers in rows that contain peaks are called peaks of a permutation. A number is called non-peak if it is not a peak. It is easy to see that the peaks of Dyck path $P$ and $\text{lift}(P)$ are labeled with the same numbers under $\Psi^{-1}$. Since we label the rows and columns that do not contain peaks from left to right with the non-peak numbers in decreasing order under the map $\Psi^{-1}$, we see that $n+1$ will be in the column of the first non-peak and that all the remaining shifts over one to the next column that does not contain a peak. Figure 8 illustrates the labeling of non-peaks of $\sigma = (8,6,9,7,4,3,2,5,1) \in S_9(123)$.

It is not difficult to see that $\sigma$ and $\text{lift}(\sigma)$ have the same Descent set in the first $n-1$ positions, and lift$(\sigma)_n$ is a descent if and only if $\sigma_n$ is a non-peak. It is very important that the lift operation preserves Descent set in the first $n-1$ positions, as we have to break descent positions to transform a permutation into an ordered set partition. Further, the word of an ordered set partition in $\mathcal{WOP}(123)$ is determined by a pair of Dyck paths, $A$ and $B$. When analyzing ordered set partition word-avoiding 123, we simply analyze the small permutations corresponding with path $A$ and $B$ and discuss a few cases at the first return position.

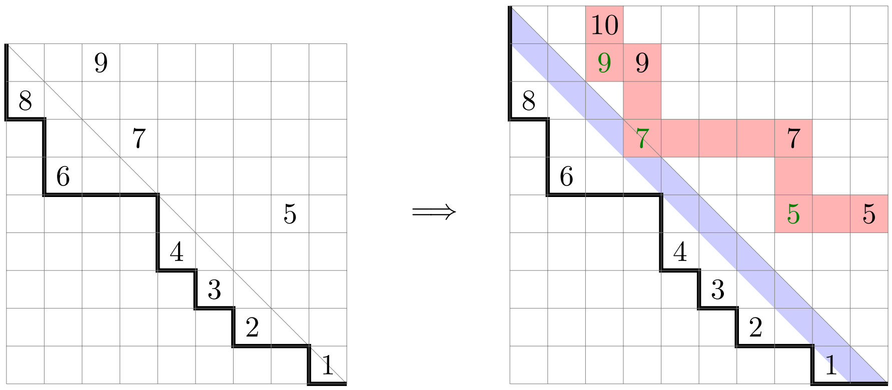


Figure 8: $\sigma = (8, 6, 9, 7, 4, 3, 2, 5, 1)$ and lift$(\sigma) = (8, 6, 10, 9, 4, 3, 2, 7, 1, 5)$

The computation of the contribution to $A(x, y, t, q_1, q_2)$ from the parts $B_1/\ldots/B_i$ which contains all the letters in $\{k+1, \ldots, n\}$ depends on the following four cases.

**Case 1.** The first return of $P$ is at the point $(1, n-1)$.

In this case, $P$ starts of $DE$ and $n$ is the first corner. This means that $w(\pi)$ starts out with $n$, $i = 1$, and $B_1 = \{n\}$. It is easy to see that in this case the contribution to $A(x, y, t, q_1, q_2)$ is $xtq_1(1+y(A(x, y, t, q_1, q_2)-1))$. That is, if $n = 1$, then we get a contribution of $xtq_1$ and otherwise, $n$ will cause a descent in $w(\pi)$ which will give a contribution of $xtq_1y(A(x, y, t, q_1, q_2) - 1)$.

**Case 2.** The first return of $P$ is at the point $(2, n-2)$.

In this case, $P$ starts of $DDEE$, $n-1$ is the first corner of $P$ and $n$ in the square $(2, n)$ so that $w(\pi)$ starts out with $(n-1)n$. Then it is either the case that $i = 2$, $B_1 = \{n-1\}$, and $B_2 = \{n\}$ or $i = 1$ and $B_1 = \{n-1, n\}$. It is easy to see that in the first case, the contribution to $A(x, y, t, q_1, q_2)$ is $x^2t^2q_1^2(1 + y(A(x, y, t, q_1, q_2) - 1)$. That is, if $n = 2$, then we get a contribution of $x^2t^2q_1^2$ and otherwise, $n$ will cause a descent in $w(\pi)$ which will give a contribution of $x^2t^2q_1^2(y(A(x, y, t, q_1, q_2) - 1)$. Similarly, in the second case the contribution to $A(x, y, t, q_1, q_2)$ is $xt^2q_2(1 + y(A(x, y, t, q_1, q_2) - 1))$. Thus the total contribution to $A(x, y, t, q_1, q_2)$ from Case 2 is

$$(x^2t^2q_1^2 + xt^2q_2)(1 + y(A(x, y, t, q_1, q_2) - 1)).$$

**Case 3.** $k < n-2$ and $k+1$ is in column $n-k-1$.
In this case, we have the situation pictured in Figure 9. Thus $w(\pi) = w_1 \ldots w_n$ where $w_{n-k-1} = k+1$ and $w_{n-k} = p$ where $k+1 < p$. It follows that either $B_i = \{k+1, p\}$ or $B_{i-1} = \{k+1\}$ and $B_i = \{p\}$. We claim that the contribution to $A(x, y, t, q_1, q_2)$ in the first case where $B_i = \{k+1, p\}$ is

$$y(A(x, y, t, q_1, q_2) - 1)xt^2q_2(1 + y(A(x, y, t, q_1, q_2) - 1)).$$

That is, the first factor of $y$ comes from the fact that there is a descent caused by the last element of $B_{i-1}$ and the first element of $B_i$ which is $k+1$. The next factor $(A(x,y,t,q_1,q_2)-1)$ comes from summing over the weights of the reductions of $B_1/\ldots/B_{i-1}$ over all possible choices of $B_1/\ldots/B_{i-1}$. The factor $xt^2q_2$ comes from $B_i$. If $B_{i+1}/\ldots/B_j$ is empty then we get a factor of 1 and, if $B_{i+1}/\ldots/B_j$ is not empty, then we get a factor of $y$ coming from the descents between the last element of $B_i$ and the first element of $B_{i+1}$ and a factor of $(A(x,y,t,q_1,q_2)-1)$ coming summing the weights over all possible choices of $B_{i+1}/\ldots/B_j$.

A similar reasoning will show that the contribution to $A(x,y,t,q_1,q_2)$ in the first case where $B_{i-1} = \{k+1\}$ and $B_i = \{p\}$ is

$$y(A(x,y,t,q_1,q_2)-1)x^2t^2q_1^2(1+y(A(x,y,t,q_1,q_2)-1)).$$

Thus the total contribution to $A(x,y,t,q_1,q_2)$ in Case 3 is

$$y(A(x,y,t,q_1,q_2)-1)(xt^2q_2+x^2t^2q_1^2)(1+y(A(x,y,t,q_1,q_2)-1)).$$

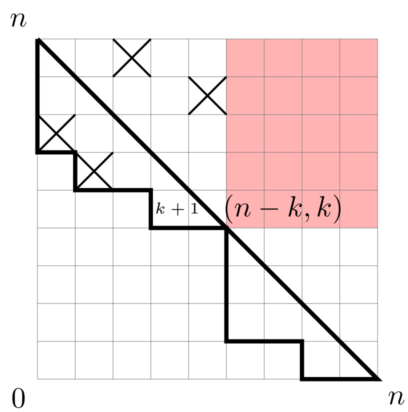


Figure 9: The situation in Case 3.

**Case 4.** $k < n-2$ and $k+1$ is in column $r$ where $r < n-k-1$.
In this case, we have the situation pictured in Figure 10. Thus $w(\pi) = w_1 \ldots w_n$ where $w_r = k+1$ and $w_{r+1}\ldots w_{n-k}$ is a decreasing sequence of length at least 2. In this situation, $B_i$ must be a singleton part $\{w_{n-k}\}$. We claim that the contribution to $A(x,y,t,q_1,q_2)$ from the ordered set partitions in Case 4 is

$$y(A(x,y,t,q_1,q_2)-1-xtq_1-xytq_1(A(x,y,t,q_1,q_2)-1))xtq_1(1+y(A(x,y,t,q_1,q_2)-1)).$$

That is, the first factor of $y$ comes from the fact that there is a descent caused by the last element of $B_{i-1}$ and the element in $B_i$. The next factor comes summing over the weights of the reductions of $B_1/\ldots/B_{i-1}$ over all possible choices of $B_1/\ldots/B_{i-1}$. It is not difficult to see that this corresponds to the sum of the weights over all non-empty ordered set partitions $\pi$ where 1 is not the last element of the word of $\pi$. Let

$$A_n(x,y,q_1,q_2) = \sum_{\pi\in\mathcal{WOP}_n(123)} x^{\ell(\pi)}y^{\mathrm{des}(w(\pi))}q_1^{\mathrm{one}(\pi)}q_2^{\mathrm{two}(\pi)}.$$

where $\mathrm{one}(\pi)$ is the number of parts of size 1 in $\pi$ and $\mathrm{two}(\pi)$ is the number of parts of size 2 in $\pi$. It is easy to see that $A_n(x,y,q_1,q_2) - xytq_1A_{n-1}(x,y,q_1,q_2)$ is the weight over all ordered set partition $\pi$ of size $n$ such that 1 is not the last element of the word of $\pi$. Thus the sum of the weights over all non-empty ordered set partitions $\pi$ where 1 is not the last element of the word of $\pi$ equals

$$\sum_{n\geq 2} t^n (A_n(x,y,q_1,q_2) - yxtq_1 A_{n-1}(x,y,q_1,q_2)) = \\ (A(x,y,t,q_1,q_2) - 1 - q_1xt) - yxtq_1(A(x,y,t,q_1,q_2) - 1).$$

Finally we get a factor of 1 if $B_{i+1}/\ldots/B_j$ is empty and a factor of $y(A(x,y,t,q_1,q_2)-1)$ over all possible choices of $B_{i+1}/\ldots/B_j$ if $B_{i+1}/\ldots/B_j$ is non-empty.

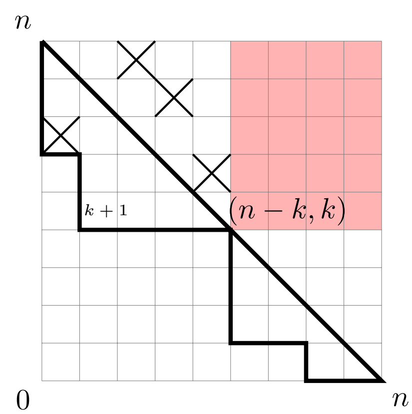


Figure 10: The situation in Case 4.

Summing the contributions from Cases 1-4, we see that

$$\begin{aligned} A(x,y,t,q_1,q_2) &= 1 + (y-1)^2(q_1xt + q_2xt^2 - q_1^2x^2t^2(y-1)) - \\ & \quad 2A(x,y,t,q_1,q_2)(y(y-1)(q_1xt + q_2xt^2 - q_1^2x^2t^2(y-1)) + \\ & \quad A(x,y,t,q_1,q_2)^2y^2(q_1xt + q_2xt^2 - q_1^2x^2t^2(y-1)). \end{aligned} \tag{14}$$

Because (14) involves both linear and quadratic terms in $x$, we can not apply the Lagrange Inversion Theorem to get an explicit formula for $\mathbb{WOP}^{\text{des}}_{123,\{1,2\}}(x,y,t,q_1,q_2)|_{x^k}$. Nevertheless, (14) gives us a quadratic equation which we can solve for $A(x,y,t,q_1,q_2)$ to prove the following theorem.

**Theorem 6.**

$$\mathbb{WOP}^{\text{des}}_{123,\{1,2\}}(x,y,t,q_1,q_2) = \frac{P(x,y,t,q_1,q_2) - \sqrt{Q(x,y,t,q_1,q_2)}}{R(x,y,t,q_1,q_2)}$$

*where*

$$\begin{aligned} P(x,y,t,q_1,q_2) &= 1 + 2y(y-1)q_1xt + 2y(y-1)q_2xt^2 - 2y(y-1)^2q_1^2x^2t^2, \\ Q(x,y,t,q_1,q_2) &= 1 - 4yq_1xt - 4yq_2xt^2 + 4(y(y-1)q_1^2x^2t^2, \ \textit{and} \\ R(x,y,t,q_1,q_2) &= 2y^2q_1xt + 2y^2q_2xt^2 - 2y^2(y-1)q_1^2x^2t^2. \end{aligned}$$

We can then use Theorem 6 to find the first few terms of $\mathbb{WOP}_{123,\{1,2\}}(x,y,t,q_1,q_2)$. That is,

$$\begin{aligned}
&\mathbb{WOP}^{\text{des}}_{123,\{1,2\}}(x,y,t,q_1,q_2) = \\
&1+(q_1x)t+(q_2x+(q_1^2+q_1^2y)x^2)t^2+ \\
&(4q_1q_2yx^2+(4q_1^3y+q_1^3y^2)x^3)t^3+ \\
&(2q_2^2yx^2+(4q_1^2q_2y+11q_1^2q_2y^2)x^3+(2q_1^4y+11q_1^4y^2+q_1^4y^3)x^4)t^4+ \\
&(15q_1q_2^2y^2x^3+(30q_1^3q_2y^2+26q_1^3q_2y^3)x^4+(15q_1^5y^2+26q_1^5y^3+q_1^5y^4)x^5)t^5+\cdots
\end{aligned}$$

Setting $y=1$ in $\mathbb{WOP}_{123,\{1,2\}}(x,y,t,q_1,q_2)$ gives us the following corollary.

**Corollary 7.**

$$1+\sum_{n\geq 1}t^n\sum_{\pi\in\mathcal{WOP}_n(123)}x^{\ell(\pi)}q_1^{\text{one}(\pi)}q_2^{\text{two}(\pi)}=\frac{1-\sqrt{1-4tx(q_1+xq_2)}}{2tx(q_1+xq_2)},$$

*and the coefficient*

$$\text{wop}_{\langle 1^{\alpha_1},2^{\alpha_2}\rangle}(123)=\frac{1}{\alpha_1+\alpha_2+1}\binom{2\alpha_1+2\alpha_2}{\alpha_1+\alpha_2}\binom{\alpha_1+\alpha_2}{\alpha_1},$$

$$\text{wop}_{n,k}(123)=\text{wop}_{\langle 1^{2k-n},2^{n-k}\rangle}(123)=\frac{1}{k+1}\binom{2k}{k}\binom{k}{n-k}.$$

*Proof.* Let $A_{123}(x,t,q_1,q_2)=\mathbb{WOP}_{123,\{1,2\}}(x,1,t,q_1,q_2)$, then the recursion becomes

$$A_{123}(x,t,q_1,q_2)=1+tq_1xA_{123}^2(x,t,q_1,q_2)+tq_2x^2A_{123}^2(x,t,q_1,q_2).$$

The formula for $A_{123}(x,t,q_1,q_2)$ is obtained by solving the quadratic equation, and the formula for the coefficients are obtained by applying Lagrange Inversion. ☐

Thus, we can calculate the number of ordered set partitions in $\mathcal{WOP}_n(123)$ with certain numbers of blocks of size 1 and size 2. Now we give a formula for the number of ordered set partitions in $\mathcal{OP}n(123)$ with a certain block size composition. In [4], Godbole, *et al.* showed that

$$\text{op}_{[b_1,\ldots,b_i,b_{i+1},\ldots,b_k]}(321)=\text{op}_{[b_1,\ldots,b_{i+1},b_i\ldots,b_k]}(321)$$

by constructing a bijective map between $\mathcal{OP}_{[b_1,\ldots,b_i,b_{i+1},\ldots,b_k]}(321)$ and $\mathcal{OP}_{[b_1,\ldots,b_{i+1},b_i\ldots,b_k]}(321)$.

For our new definition of pattern avoidance, We prove the similar result that the order of block sizes in block size composition won't make a difference to $\text{wop}_{[b_1,\ldots,b_k]}(123)$, and then we calculate the formula for $\text{wop}_{[b_1,\ldots,b_k]}(123)$.

**Theorem 8.** *We have*

$$\text{wop}_{[b_1,\ldots,b_i,b_{i+1},\ldots,b_k]}(123)=\text{wop}_{[b_1,\ldots,b_{i+1},b_i\ldots,b_k]}(123)$$

*and*

$$\text{wop}_{[b_1,\ldots,b_i,b_{i+1},\ldots,b_k]}(321)=\text{wop}_{[b_1,\ldots,b_{i+1},b_i\ldots,b_k]}(321).$$

*Proof.* The second equation is included in the bijection constructed by Godbole, *et al.* that

$$\begin{aligned}\mathrm{wop}_{[b_1,\ldots,b_i,b_{i+1},\ldots,b_k]}(321) &= \mathrm{op}_{[b_1,\ldots,b_i,b_{i+1},\ldots,b_k]}(321)\\ &= \mathrm{op}_{[b_1,\ldots,b_{i+1},b_i,\ldots,b_k]}(321)\\ &= \mathrm{wop}_{[b_1,\ldots,b_{i+1},b_i,\ldots,b_k]}(321).\end{aligned}$$

For the first equation, we prove by a bijection.

For a block size composition $B = [b_1, \ldots, b_i, b_{i+1}, \ldots, b_k]$, since we are considering the 123-avoiding ordered set partitions, all the blocks are of size either 1 or 2. We only need to show equality in 2 cases.

(1) If $b_i = b_{i+1} = 1$ or 2, then $\mathrm{op}_{[b_1,\ldots,b_i,b_{i+1},\ldots,b_k]}(123)$ and $\mathrm{op}_{[b_1,\ldots,b_{i+1},b_i,\ldots,b_k]}(123)$ are exactly the same enumerations.

(2) If $b_i \neq b_{i+1}$, then without loss of generality, we suppose $b_i = 1$ and $b_{i+1} = 2$. We show that there is a bijective map between $\mathcal{WOP}[b_1, \ldots, 1, 2, \ldots, b_k](123)$ and $\mathcal{WOP}[b_1, \ldots, 2, 1, \ldots, b_k](123)$. We suppose the 3 integers filled in blocks $b_i$ and $b_{i+1}$ are $a_1 < a_2 < a_3$. Since there is no 123 pattern-match, there are only 2 possible fillings for both $[\ldots, 1, 2, \ldots]$ and $[\ldots, 2, 1, \ldots]$ cases. They are $a_2/a_1a_3$ and $a_3/a_1a_2$ for $[\ldots, 1, 2, \ldots]$, $a_2a_3/a_1$ and $a_1a_3/a_2$ for $[\ldots, 2, 1, \ldots]$. We construct a map, as showed in Figure 11, mapping $a_2/a_1a_3$ to $a_2a_3/a_1$ and $a_3/a_1a_2$ to $a_1a_3/a_2$.

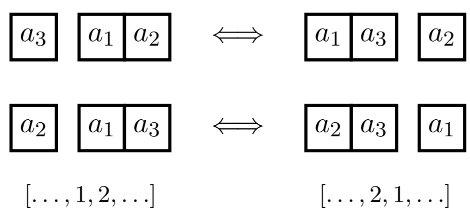


Figure 11: Bijection between $\mathcal{WOP}_{[b_1,\ldots,1,2,\ldots,b_k]}(123)$ and $\mathcal{WOP}_{[b_1,\ldots,2,1,\ldots,b_k]}(123)$.

It is not difficult to check that the map is bijective and preserves 123-avoiding condition. Thus $\mathrm{wop}_{[b_1,\ldots,b_i,b_{i+1},\ldots,b_k]}(321) = \mathrm{wop}_{[b_1,\ldots,b_{i+1},b_i\ldots,b_k]}(321)$. □

The formula about $\mathrm{wop}_{[b_1,\ldots,b_i,b_{i+1},\ldots,b_k]}(123)$ follows the bijection.

**Theorem 9.** *For any composition* $[b_1, \ldots, b_k](b_i \in \{1, 2\})$*, we have*

$$\mathrm{wop}_{[b_1,\ldots,b_k]}(123) = C_k,$$

*here* $C_k = \frac{1}{k+1}\binom{2k}{k}$ *is the* $k^{\text{th}}$ *Catalan number.*

*Proof.* Let $\alpha_1$ be the number of 1's and $\alpha_2$ be the number of 2's in $[b_1, \ldots, b_k]$. From Corollary 7, we have the formula

$$\mathrm{wop}_{\langle 1^{\alpha_1}, 2^{\alpha_2}\rangle}(123) = \frac{1}{\alpha_1 + \alpha_2 + 1}\binom{2\alpha_1 + 2\alpha_2}{\alpha_1 + \alpha_2}\binom{\alpha_1 + \alpha_2}{\alpha_1}.$$

Since the order of block sizes won't make a difference to $\mathrm{wop}_{[b_1,\ldots,b_k]}(123)$ and there are $\binom{\alpha_1+\alpha_2}{\alpha_1}$ ways to permute the block size, we have

$$\mathrm{wop}_{[b_1,\ldots,b_k]}(123) = \frac{\frac{1}{\alpha_1+\alpha_2+1}\binom{2\alpha_1+2\alpha_2}{\alpha_1+\alpha_2}\binom{\alpha_1+\alpha_2}{\alpha_1}}{\binom{\alpha_1+\alpha_2}{\alpha_1}} = \frac{1}{\alpha_1+\alpha_2+1}\binom{2\alpha_1+2\alpha_2}{\alpha_1+\alpha_2} = \frac{1}{k+1}\binom{2k}{k} = C_k.$$

□

Setting $y = q_1 = q_2 = 1$ in $\mathbb{WOP}_{123,\{1,2\}}(x,y,t,q_1,q_2)$ gives us the following corollary.
**Corollary 10.**

$$1 + \sum_{n\geq 1} t^n \sum_{\pi \in \mathcal{WOP}_n(123)} x^{\ell(\pi)} = \frac{1-\sqrt{1-4tx-4t^2x}}{2(xt+xt^2)}.$$

The initial terms of the series $1 + \sum_{n\geq 1} t^n \sum_{\pi \in \mathcal{WOP}_n(123)} x^{\ell(\pi)}$ are

$$\begin{aligned}
&1 + xt + (x+2x^2)t^2 + (4x^2+5x^3)t^3 + (2x^2+15x^3+14x^4)t^4 + \\
&(15x^3+56x^4+42x^5)t^5 + (5x^3+84x^4+210x^5+132x^6)t^6 + \\
&(56x^4+420x^5+792x^6+429x^7)t^7 + \\
&(14x^4+420x^5+1980x^6+3003x^7+1430x^8)t^8 + \\
&(210x^5+2640x^6+9009x^7+11440x^8+4862x^9)t^9 + \cdots
\end{aligned}$$

We pause to make some observations about some special cases of elements of $\mathcal{WOP}_n(123)$. First consider the case of ordered set partitions in $\mathcal{WOP}_n(123)$ where every part has size 1. In this case, we are just considering the generating function of $y^{\mathrm{des}(\sigma)}$ over all 123-avoiding permutations. We can obtain this generating function from $\mathbb{WOP}_{123,\{1,2\}}(x,y,t,q_1,q_2)$ by setting $x$ equal to $1/x$, $t$ equal to $tx$, and then setting $x=0$. We carried out these steps in Mathematica and obtained the following corollary which was first proved by Barnabei, Bonetti, and Silimbani [1].
**Corollary 11.**

$$1 + \sum_{n\geq 1} t^n \sum_{\sigma \in S_n(123)} y^{\mathrm{des}(\sigma)} = \frac{-1-2ty(y-1)+2t^t y(y-1)^2+\sqrt{1-4ty-4t^2y(y-1)}}{2ty^2(-1+t(y-1))}.$$

The first few terms $1 + \sum_{n\geq 1} t^n \sum_{\sigma \in S_n(123)} y^{\mathrm{des}(\sigma)}$ are

$$\begin{aligned}
&1 + t + (1+y)t^2 + (4y+y^2)t^3 + (2y+11y^2+y^3)t^4 + \\
&(15y^2+26y^3+y^4)t^5 + (5y^2+69y^3+57y^4+y^5)t^6 + \\
&(56y^3+252y^4+120y^5+y^6)t^7 + (14y^3+364y^4+804y^5+247y^6+y^7)t^8 + \\
&(201y^4+1880y^5+2349y^6+502y^7+y^8)t^9 + \cdots.
\end{aligned}$$

We can do a similar computation starting with the generating function $\mathbb{WOP}^{\mathrm{des}}_{132}(x,y,t)$ to obtain the following corollary.
**Corollary 12.** *For any* $\alpha \in \{132, 231, 312, 213\}$,

$$1 + \sum_{n\geq 1} t^n \sum_{\sigma \in S_n(\alpha)} y^{\mathrm{des}(\sigma)} = \frac{1+t(y-1)-\sqrt{1+t^2(y-1)^2-2t(y+1)}}{2yt}.$$

The first few terms $1+\sum_{n\geq 1} t^n \sum_{\sigma\in S_n(132)} y^{\mathrm{des}(\sigma)}$ are

$$\begin{aligned}
&1+t+(1+y)t^2+(1+3y+y^2)t^3+(1+6y+6y^2+y^3)t^4+\\
&(1+10y+20y^2+10y^3+y^4)t^5+(1+15y+50y^2+50y^3+15y^4+y^5)t^6+\\
&(1+21y+105y^2+175y^3+105y^2+21y^5+y^6)t^7+\\
&(1+28y+196y^2+490y^3+490y^4+196y^5+28y^6+y^7)t^8+\\
&(1+36y+336y^2+1176y^3+1764y^4+1176y^5+336y^6+36y^7+y^8)t^9+\cdots.
\end{aligned}$$

In this case, the coefficients are the coefficients of the triangle of the Narayana numbers $T(n,k)=\frac{1}{k}\binom{n}{k-1}\binom{n-1}{k-1}$ which is entry A001263 in the OEIS [11].

The final generating function that we shall consider in this section is $\mathbb{WOP}_{321}^{\mathrm{des}}(x,y,t)$. Since a permutation $\sigma$ is 321-avoiding if and only if its reverse $\sigma^r$ is 123-avoiding, we shall again appeal to the bijection $\Psi$ of Deutsch and Elizalde between 123-avoiding permutations and Dycks paths and classify the ordered set partitions $\delta$ whose word avoids 321 by $\Psi(w(\delta))$. The main difference in this case is that we obtain the permutation $w(\delta)$ by reading the elements in the diagram from right to left, rather from left to right, and we classify the ordered set partitions by the last return of $\Psi(w(\delta))$. In this situation, we have two cases for any $\delta\in\mathcal{WOP}_n(321)$.

**Case 1.** The last return of $\Psi(w(\delta))$ is at position $(n-1,1)$ in which case $\sigma$ starts with 1.

In this case, 1 can not be part of an occurrence of 321 in the word of the ordered set partition. Thus either 1 is in a part by itself in which case we get a contribution of $xt\mathbb{WOP}_{321}^{\mathrm{des}}(x,y,t)$ to $\mathbb{WOP}_{321}^{\mathrm{des}}(x,y,t)$ or 1 is part of the first part of the ordered set partition arising from the part of the ordered set partition above and to the left of 1 which will give a contribution of $t(\mathbb{WOP}_{321}^{\mathrm{des}}(x,y,t)-1)$ to $\mathbb{WOP}_{321}^{\mathrm{des}}(x,y,t)$. Thus the total contribution to $\mathbb{WOP}_{321}^{\mathrm{des}}(x,y,t)$ of the ordered set partitions whose word avoids 321 and starts with 1 is

$$xt\mathbb{WOP}_{321}^{\mathrm{des}}(x,y,t)+t(\mathbb{WOP}_{321}^{\mathrm{des}}(x,y.t)-1).$$

**Case 2.** Either $\Psi(w(\delta))$ has no return or the last return is at position $(n-k,k)$ where $k>1$.

Let us first consider the cases of ordered set partitions $\delta\in\mathcal{WOP}_n(321)$ such that $\Psi(w(\delta))$ hits the diagonal only at $(0,n)$ and $(n,0)$ and $n\geq 2$. For such ordered set partitions, we have two subcases.

**Subcase 2.1** The second element of $w(\delta)$ equals 1.

In this case, suppose that $w(\delta)=w_1\dots w_n$ where $w_2=1$. Then we have the situation pictured in Figure 12. In this case, since $w_1>w_2=1$, it must be that case that $w_1$ is in a part by itself so that it will contribute a factor of $xyt$ to the weight of $\delta$. If we remove the row and column containing $w_1$ and keep the same outside corner squares, and possibly re-label the $\times$ in the columns with no outside corner squares by having the $\times$ in those columns decrease, reading from left to right, we will obtain an arbitrary ordered set partition $\pi\in\mathcal{WOP}_{n-1}(321)$ such that $w(\pi)$ starts with 1. Hence the ordered set partitions in this subcase will contribute a factor of

$$xyt(xt\mathbb{WOP}_{321}^{\mathrm{des}}(x,y,t)+t(\mathbb{WOP}_{321}^{\mathrm{des}}(x,y,t)-1))$$

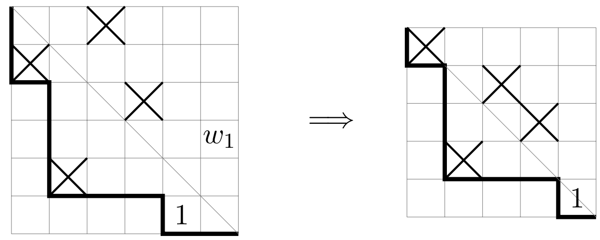


Figure 12: Ordered set partitions in Subcase 2.1.

to $\mathbb{WOP}_{321}^{\text{des}}(x,y,t)$.

**Subcase 2.2**

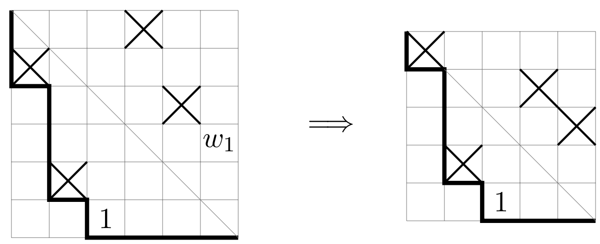


Figure 13: Ordered set partitions in Subcase 2.2.

In this case, suppose that $w(\delta) = w_1 \ldots w_n$ where $w_i = 1$ for $i > 2$. Then we have the situation pictured in Figure 13. In this case, since $w_1 < w_2 < \cdots < w_{i-1} > w_i = 1$, it must be that case that $w_i$ starts a new part in $\delta$. If we remove the row and column containing $w_1$ and keep the same outside corner squares, are possibly re-label the $\times$ in the columns with no outside corner squares by having the $\times$ in those column decrease, reading from left to right, we will obtain an arbitrary ordered set partition $\pi \in \mathcal{WOP}_{n-1}(321)$ such that $w(\pi)$ does not start with 1. The sum of the weights of the ordered set partitions $\pi$ such that $w(\pi)$ does not start with 1 is

$$\mathbb{WOP}_{321}^{\text{des}}(x,y,t) - 1 - xyt\mathbb{WOP}_{321}^{\text{des}}(x,y,t) - t(\mathbb{WOP}_{321}^{\text{des}}(x,y,t) - 1).$$

Then $w_1$ is either in a part by itself in which case it contributes a factor of $xt$ or is in the part with $w_2$ in which case it contributes a factor of $t$. Hence the ordered set partitions in this subcase will contribute a factor of

$$(xt+t)(\mathbb{WOP}_{321}^{\text{des}}(x,y,t) - 1 - xyt\mathbb{WOP}_{321}^{\text{des}}(x,y,t) - t(\mathbb{WOP}_{321}^{\text{des}}(x,y,t) - 1))$$

to $\mathbb{WOP}_{321}^{\text{des}}(x,y,t)$.

Thus the ordered set partitions $\delta \in \mathcal{WOP}_n(321)$ such that $\Psi(w(\delta))$ hits the diagonal only at $(0,n)$ and $(n,0)$ and $n \geq 2$ contribute a factor of

$$\begin{aligned} NR(x,y,t) = xyt(xt\mathbb{WOP}_{321}^{\text{des}}(x,y,t) + t(\mathbb{WOP}_{321}^{\text{des}}(x,y,t) - 1)) + \\ (xt+t)(\mathbb{WOP}_{321}^{\text{des}}(x,y,t) - 1 - xyt\mathbb{WOP}_{321}^{\text{des}}(x,y,t) - t(\mathbb{WOP}_{321}^{\text{des}}(x,y,t) - 1)) \end{aligned} \quad (15)$$

to $\mathbb{WOP}_{321}^{\text{des}}(x,y,t)$.

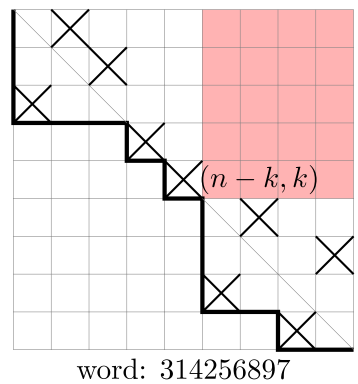


Figure 14: The general situation in Case 2.

Now consider in the general case in Case 2 where the last return is at $(n-k,k)$ where $1 \leq k < n-1$. This situation is pictured in Figure 14. Because we fill the columns which do not have outside corner cells in a decreasing manner, reading from left to right, it is easy to see that there can be no $\times$ in the cells of the red area in Figure 14. This means that the $\times$ corresponding to $1,\ldots,k$ must all be in the bottom $k \times k$ squares. What we don't know is how the final increasing sequence of the elements $1,\ldots,k$ in $w(\delta)$ union of the initial increasing sequence of the remaining elements break up into parts in $\delta$. For example, in Figure 14, $k=4$ and the last increasing sequence of the elements $1,\ldots,5$ in $w(\delta)$ is 2 and the initial increasing sequence of the remaining elements is $6,7,9,10$. Then we have two cases. The first case is when there is no overlap between the parts containing $1,\ldots,k$ and the remaining parts. In this case, we get a contribution of $NR(x,y,t)(\mathbb{WOP}_{321}^{\mathrm{des}}(x,y,t)-1)$ to $\mathbb{WOP}_{321}^{\mathrm{des}}(x,y,t)$. If there is an overlap, then we need to remove the $x$ corresponding to the last part in the generating function $NR(x,y,t)$ so that we would get a contribution of $\frac{1}{x}D(x,y,t)(\mathbb{WOP}_{321}^{\mathrm{des}}(x,y,t)-1)$.

It follows that the total contribution to $\mathbb{WOP}_{321}^{\mathrm{des}}(x,y,t)$ from the ordered set partitions $\delta \in \mathcal{WOP}(321)$ in Case 2 is

$$NR(x,y,t) + (1-\frac{1}{x})NR(x,y,t)(\mathbb{WOP}_{321}^{\mathrm{des}}(x,y,t)-1). \tag{16}$$

Hence we see that

$$\begin{aligned}\mathbb{WOP}_{321}^{\mathrm{des}}(x,y,t) &= 1 + xt\mathbb{WOP}_{321}^{\mathrm{des}}(x,y,t) + t(\mathbb{WOP}_{321}^{\mathrm{des}}(x,y,t)-1) + \\ &\quad NR(x,y,t) + (1-\frac{1}{x})D(x,y,t)(\mathbb{WOP}_{321}^{\mathrm{des}}(x,y,t)-1)\end{aligned}$$

where $NR(x,y,t)$ is defined as in (15). This is a quadratic equation in $\mathbb{WOP}_{321}^{\mathrm{des}}(x,y,t)$ which we can solve to obtain the following theorem.

**Theorem 13.**

$$\mathbb{WOP}_{321}^{\mathrm{des}}(x,y,t) = \frac{2t(1+x)(1+t((x(y-1)-1)) - x - \sqrt{x^2(1-4t(1+x))(1+t((x(y-1)-1))}}{2t(1+x)^2(1+t((x(y-1)-1))} \tag{17}$$

The first few terms of $\mathbb{WOP}^{\text{des}}_{321}(x,y,t)$ are as follows.

$$\mathbb{WOP}^{\text{des}}_{321}(x,y,t) = 1 + xt + (x + (1+y)x^2)t^2 +$$
$$(x + (2+4y)x^2 + (1+4y)x^3)t^3 +$$
$$(x + (3+11y)x^2 + (3+22y+y^2)x^3 + (1+11y+2y^2)x^4)t^4 +$$
$$(x + (4+26y)x^2 + (6+78y+15y^2)x^3 + (4+78y+30y^2)x^4 + (1+26y+15y^2)x^5)t^5 + \cdots$$

Setting $y = 1$ in (17), we obtain the following corollary which recovers the result of Chen, Dai, and Zhou [2].

**Corollary 14.**

$$\mathbb{WOP}^{\text{des}}_{321}(x,1,t) = \frac{2t^2(1+x) - 2t(1+x) - x - x\sqrt{1-4t(1+x)+4t^2(1+x)}}{2t(t-1)(1+x)}. \tag{18}$$

The recursion that we used to compute $\mathbb{WOP}^{\text{des}}_{321}(x,y,t)$ does not allow us to control the size of the parts of the ordered set partitions $\pi \in \mathcal{WOP}_n(321)$ so that we have not been able to compute generating functions of the form $\mathbb{WOP}^{\text{des}}_{\{b_1,\ldots,b_k\}}(x,y,t,q_1,\ldots,q_k)$ in general.

# 4 Generating functions for min-descents

Based on the analysis in Section 2, we need to study the following 5 kinds of generating functions,

$$\begin{array}{ccc}
\mathbb{WOP}^{\text{mindes}}_{213}(x,y,t) & = & \mathbb{WOP}^{\text{mindes}}_{312}(x,y,t), \\
\mathbb{WOP}^{\text{mindes}}_{132}(x,y,t) & , & \mathbb{WOP}^{\text{mindes}}_{231}(x,y,t), \\
\mathbb{WOP}^{\text{mindes}}_{123}(x,y,t) & , & \mathbb{WOP}^{\text{mindes}}_{321}(x,y,t).
\end{array}$$

We are able to solve the functions $\mathbb{WOP}^{\text{mindes}}_{132}(x,y,t)$ and $\mathbb{WOP}^{\text{mindes}}_{231}(x,y,t)$, and write the functions $\mathbb{WOP}^{\text{mindes}}_{213}(x,y,t) = \mathbb{WOP}^{\text{mindes}}_{312}(x,y,t)$, $\mathbb{WOP}^{\text{mindes}}_{123}(x,y,t)$ and $\mathbb{WOP}^{\text{mindes}}_{321}(x,y,t)$ as roots of polynomial equations respectively.

## 4.1 The function $\mathbb{WOP}^{\text{mindes}}_{132}(x,y,t)$

As we observed in Section 2,

$$\mathbb{WOP}^{\text{des}}_{132}(x,y,t) = \mathbb{WOP}^{\text{mindes}}_{132}(x,y,t),$$

Thus we have the following theorem.
**Theorem 15.**

$$\mathbb{WOP}^{\text{mindes}}_{132}(x,y,t) = \mathbb{WOP}^{\text{mindes}}_{231}(x,y,t) =$$
$$\frac{(1+2yt+xyt-t-tx) - \sqrt{(1+2yt+xyt-t-tx)^2 - 4(1-t+ty)(t(y+xy))}}{2t(y+yx)}$$

*and*

$$\sum_{\pi \in \mathcal{WOP}_{n,k}(132)} y^{\text{mindes}(\pi)} = \frac{1}{k}\binom{n-1}{k-1}\sum_{j=0}^{k-1}\binom{k}{j}\binom{n}{k-1-j}y^{k-1-j}.$$

## 4.2 The function $\mathbb{WOP}_{231}^{\text{mindes}}(x,y,t)$

Next consider $\mathbb{WOP}_{231}^{\text{mindes}}(x,y,t)$. Let

$$C_n(x,y) = \sum_{\pi \in \mathcal{WOP}_n(231)} x^{\ell(\pi)} y^{\text{mindes}(\pi)}.$$

We can classify the $\pi = B_1/\ldots/B_k \in \mathcal{WOP}_n(231)$ by the position $i$ of $n$ in the word of $\pi$. Assume $n \geq 2$.

**Case 1.** $i = 1$.
In this case $w(\pi)$ starts with $n$ which means that $n$ must be in a part by itself so that $B_1 = \{n\}$. Then $B_1$ will contribute a factor of $xy$ since it will automatically cause a min-descent with $B_2$. Thus the ordered set partitions $\pi \in \mathcal{WOP}_n(231)$ in Case 1 will contribute $xyC_{n-1}(x,y)$ to $C_n(x,y)$.

**Case 2.** $i = n$.
In this case $w(\pi)$ ends with $n$. Now if $n$ is in a part by itself, then $B_k = \{n\}$. Thus there will not be a min-descent between $B_{k-1}$ and $B_k$. Hence we will get a contribution of $C_{n-1}(x,y)$ in this case. If $n \in B_k$ where $|B_k| \geq 2$, then we can simply remove $n$ from $B_k$ and obtain an ordered set partition in $\mathcal{WOP}_n(231)$ with the same number of parts and the same number of min-descents. Thus the ordered set partitions $\pi \in \mathcal{WOP}_n(231)$ in Case 2 will contribute $(1+x)C_{n-1}(x,y)$ to $C_n(x,y)$.

**Case 3.** $2 \leq i \leq n-1$.
In this case, $n$ must be the last element in some part $B_i$. Because $w(\pi)$ is 231-avoiding, it must be the case that all the elements in $B_1/\ldots/B_i - \{n\}$ must be less than all the elements in $B_{i+1}/\ldots/B_k$. If $B_i = \{n\}$, then $B_i$ will contribute a factor of $xy$ since $B_i$ will cause a min-descent with $B_{i+1}$. Our choices over all possibilities of $B_1/\ldots/B_{i-1}$ will contribute a factor of $C_{i-1}(x,y)$ and our choices over all possible choices of $B_{i+1}/\ldots/B_k$ will contribute a factor of $C_{n-i}(x,y)$. Thus we will get a contribution of $xyC_{i-1}(x,y)C_{n-1}(x,y)$ in this case. If $|B_i| \geq 2$, then we can eliminate $n$ from $B_i$. Our choices over all possibilities of $B_1/\ldots/B_i - \{n\}$ will contribute a factor of $C_{i-1}(x,y)$ and our choices over all possible choices of $B_{i+1}/\ldots/B_k$ will contribute a factor of $C_{n-i}(x,y)$. Hence we will get a contribution of $C_{i-1}(x,y)C_{n-1}(x,y)$ in this situation. Thus the ordered set partitions $\pi \in \mathcal{WOP}_n(231)$ in Case 3 will contribute $(1+xy)C_{i-1}(x,y)C_{n-i}(x,y)$ to $C_n(x,y)$.

It follows that for $n \geq 2$,

$$C_n(x,y) = (1+x+xy)C_{n-1}(x,y) + \sum_{i=2}^{n-1}(1+xy)C_{i-1}(x,y)C_{n-i}(x,y).$$

Hence,

$$\begin{aligned}
\mathbb{WOP}_{231}^{\text{mindes}}(x,y,t) &= 1 + xt + \sum_{n\geq 2} C_n(x,y)t^n \\
&= 1 + xt + (1+x+xy)t\sum_{n\geq 2} C_{n-1}(x,y)t^{n-1} + (1+xy)t\sum_{n\geq 2}\sum_{k=2}^{n-1} C_{k-1}(x,y)C_{n-k}(x,y) \\
&= 1 + xt + (1+x+xy)t(\mathbb{WOP}_{231}^{\text{mindes}}(x,y,t)-1) + (1+xy)t(\mathbb{WOP}_{231}^{\text{mindes}}(x,y,t)-1)^2.
\end{aligned}$$

This gives us a quadratic equation in which we can solve to prove the following theorem.

**Theorem 16.**

$$\mathbb{WOP}_{231}^{\text{mindes}}(x,y,t) = \frac{1+t-tx+txy-\sqrt{(1+t-tx+txy)^2-4(t+txy)}}{2(t+txy)}.$$

Below are the first few terms of $\mathbb{WOP}_{231}^{\text{mindes}}(x,y,t)$.

$$\begin{aligned}
&\mathbb{WOP}_{231}^{\text{mindes}}(x,y,t) = \\
&1+xt+(x+(1+y)x^2)t^2+(x+(3+2y)x^2+(1+3y+y^2))t^3+ \\
&(x+(6+3y)x^2+(6+12y+3y^2)x^3+(1+6y+6y^2+y^3)x^4)t^4+ \\
&\big(x+(10+4y)x^2+(20+30y+6y^2)x^3+(10+40y+30y^2+4y^3)x^4+ \\
&(1+10y+20y^2+10y^3+y^4)x^5\big)\,t^5+\cdots
\end{aligned}$$

### 4.3 The functions $\mathbb{WOP}_{213}^{\text{mindes}}(x,y,t) = \mathbb{WOP}_{312}^{\text{mindes}}(x,y,t)$

Note that the set $\mathcal{WOP}(213)$ is in bijection with $\mathcal{WOP}(132)$ by the action reverse-complement, so we can work on the set $\mathcal{WOP}(132)$ and track the descent of the maximum number of blocks, or maxdes of ordered set partitions in $\mathcal{WOP}(132)$ to compute the function $\mathbb{WOP}_{213}^{\text{mindes}}(x,y,t)$.

We denote generating function that tracks maxdes of $\mathcal{WOP}(132)$ by $\mathbb{WOP}_{132}^{\text{maxdes}}(x,y,t)$. We shall again classify the ordered set partitions $\pi \in \mathcal{WOP}_n(132)$ by size of the last part and we will use the notation from Figure 3. Now suppose that $D(x,y,t) = \mathbb{WOP}_{132}^{\text{maxdes}}(x,y,t)$. In this case, we get a factor of $xt^r$ from the last part $\{a_1,\ldots,a_r\}$. Next we have to analyze when the last part from any $A_i$ will cause a max-descent in $\pi$. Let $s$ be the smallest index i such that $A_i$ is non-empty. If $s = r+1$, then there will be a max-descent from the last part of $A_{r+1}$ to $\{a_1,\ldots,a_r\}$ so that we would get a factor of $y(D(x,y,t)-1)$. If $s \leq r$, then the last part of $A_s$ will not create a max-descent with $\{a_1,\ldots,a_r\}$ so it will just contribute a factor of $(D(x,y,t)-1)$. However, each non-empty $A_j$ with $j > s$ will create a max-descent between the last part of $A_j$ and the first part of the next non-empty $A_i$ which immediately follows it so that each such $A_j$ will contribute a factor of $1+y(D(x,y,t)-1)$. Thus $D = D(x,y,t)$ satisfies the following recursive relation.

$$\begin{aligned}
D(x,y,t) &= 1+\sum_{r\geq 1} xt^r \left((1+y(D-1)+\sum_{s=1}^{r}(D-1)(1+y(D-1))^{r+1-s}\right) \\
&= 1+x(1+y(D-1))\sum_{r\geq 1} t^r \left(1+(D-1)\sum_{s=1}^{r}(1+y(D-1))^{r-s}\right) \\
&= 1+x(1+y(D-1))\sum_{r\geq 1} t^r \left(1+(D-1)\frac{(1+y(D-1))^r-1}{(1+y(D-1))-1}\right) \\
&= 1+x(1+y(D-1))\sum_{r\geq 1} t^r \left(1-\frac{(1+y(D-1))^r-1}{y}\right) \\
&= 1+x\frac{(1+y(D-1))}{y}\sum_{r\geq 1} t^r \left(y-1+(1+y(D-1))^r\right) \\
&= 1+x\frac{(1+y(D-1))}{y}\left(\frac{t(y-1)}{1-t}+\frac{t(1+y(D-1))}{1-t(1+y(D-1))}\right) \\
&= 1+\frac{tx}{y}(1+y(D-1))\left(\frac{(y-1)}{1-t}+\frac{(1+y(D-1))}{1-t(1+y(D-1))}\right). \qquad (19)
\end{aligned}$$

Clearing the fractions gives a quadric function in $D$ which we can solve to show that

$$D(x,y,t) = \frac{P(x,y,t)) - \sqrt{(Q(x,y,t)}}{R(x,y,t)} \tag{20}$$

where

$$\begin{aligned} P(x,y,t) &= 1 - 2t + t^2 - tx + 2ty - 2t^2y + txy + 2t^2xy - 2t^2xy^2 \\ Q(x,y,t) &= 1 - 4t + 6t^2 - 4t^3 + t^4 - 2tx + 4t^2x - 2t^3x + t^2x^2 - 2txy + \\ & \quad 4t^2xy - 2t^3xy - 2t^2x^2y + t^2x^2y^2, \text{ and} \\ R(x,y,t) &= 2(ty - t^2y + txy - t^2xy^2). \end{aligned}$$

If we let $f(x,y,t) = D(x,y,t) - 1$, then (19) gives that

$$f(x,y,t) = x\frac{t}{y}(1+yf)\left(\frac{y-1}{1-t} + \frac{1+yf}{1-t(1+yf)}\right). \tag{21}$$

The Lagrange Inversion Theorem implies that the coefficient of $x^k$ in $f(x,y,t)$, $f(x,y,t)|_{x^k}$, is given by

$$f(x,y,t)|_{x^k} = \frac{1}{k}\Delta(x)^k|_{x^{k-1}},$$

where

$$\Delta(x) = \frac{t}{y}(1+yx)\left(\frac{y-1}{1-t} + \frac{1+yx}{1-t(1+yx)}\right).$$

Thus

$$\begin{aligned} & f(x,y,t)|_{x^kt^n} \\ = \;& \frac{1}{k}\frac{t^k}{y^k}(1+yx)^k \sum_{a=0}^{k} \binom{k}{a} \frac{(y-1)^{k-a}}{(1-t)^{k-a}} \frac{(1+xy)^a}{(1-t(1+xy))^a}|_{x^{k-1}t^n} \\ = \;& \frac{1}{k}\frac{1}{y^k} \sum_{a=0}^{k} \binom{k}{a} \frac{(y-1)^{k-a}}{(1-t)^{k-a}} \frac{(1+xy)^{k+a}}{(1-t(1+xy))^a}|_{x^{k-1}t^{n-k}}. \end{aligned}$$

By Newton's Binomial Theorem

$$\begin{aligned} \frac{1}{(1-t)^{k-a}} &= \sum_{u\geq 0} \binom{k-a+u-1}{u} t^u \text{ and} \\ \frac{1}{(1-t(1+xy))^a} &= \sum_{v\geq 0} \binom{a+v-1}{v} t^v((1+xy)^v. \end{aligned}$$

It follows that

$$\begin{aligned} & f(x,y,t)|_{x^kt^n} = \\ & \frac{1}{k}\frac{1}{y^k}\sum_{a=0}^{k}\sum_{v=0}^{n-k}\binom{k}{a}\binom{a+v-1}{v}\binom{k-a+(n-k-v)-1}{n-k-v}(y-1)^{k-a}(1+xy)^{k+a+v}|_{x^{k-1}} = \\ & \frac{1}{k}\frac{1}{y^k}\sum_{a=0}^{k}\sum_{v=0}^{n-k}\binom{k}{a}\binom{a+v-1}{v}\binom{k-a+(n-k-v)-1}{n-k-v}\binom{k+a+v}{k-1}(y-1)^{k-a}y^{k-1} = \\ & \frac{1}{k}\frac{1}{y}\sum_{a=0}^{k}\sum_{v=0}^{n-k}\binom{k}{a}\binom{a+v-1}{v}\binom{k-a+(n-k-v)-1}{n-k-v}\binom{k+a+v}{k-1}(y-1)^{k-a}. \end{aligned}$$

Thus we have the following theorem.
**Theorem 17.**

$$\mathbb{WOP}_{213}^{\text{mindes}}(x,y,t) = \mathbb{WOP}_{312}^{\text{mindes}}(x,y,t) = \mathbb{WOP}_{132}^{\text{maxdes}}(x,y,t) = \frac{P(x,y,t) - \sqrt{(Q(x,y,t)}}{R(x,y,t)}, \quad (22)$$

*where*

$$\begin{aligned} P(x,y,t) &= 1 - 2t + t^2 - tx + 2ty - 2t^y + txy + 2t^2xy - 2t^2xy^2, \\ Q(x,y,t) &= 1 - 4t + 6t^2 - 4t^3 + t^4 - 2tx + 4t^2x - 2t^3x + t^2x^2 - 2txy + \\ &\quad 4t^2xy - 2t^3xy - 2t^2x^2y + t^2x^2y^2, \text{ and} \\ R(x,y,t) &= 2(ty - t^2y + txy - t^2xy^2), \end{aligned}$$

*and*

$$\sum_{\pi \in \mathcal{WOP}_{n,k}(213)} y^{\text{mindes}(\pi)} = \frac{1}{k}\frac{1}{y}\sum_{a=0}^{k}\sum_{v=0}^{n-k}\binom{k}{a}\binom{a+v-1}{v}\binom{k-a+(n-k-v)-1}{n-k-v}\binom{k+a+v}{k-1}(y-1)^{k-a}. \quad (23)$$

Below are the first few terms of $\mathbb{WOP}_{213}^{\text{mindes}}(x,y,t)$.

$$\begin{aligned} &\mathbb{WOP}_{213}^{\text{mindes}}(x,y,t) = \\ &1 + xt + (x + (1+y)x^2)t^2 + (x + (3+2y)x^2 + (1+3y+y^2))t^3 + \\ &(x + (6+3y)x^2 + (5+13y+3y^2)x^3 + (1+6y+6y^2+y^3)x^4)t^4 + \\ &\big(x + (10+4y)x^2 + (15+35y+6y^2)x^3 + (7+39y+34Y62+4y^3)x^4 + \\ &(1+10y+20y^2+10y^3+y^4)x^5\big)\, t^5 + \cdots. \end{aligned}$$

We can compute the limit as $y \to 0$ of $\mathbb{WOP}_{213}^{\text{mindes}}(x,y,t)$ to obtain the generating function of ordered set partitions in $\mathcal{WOP}_n(213)$ which have no min-descents. In this case, we obtain the following corollary.
**Corollary 18.**

$$1 + \sum_{n\geq 1} t^n \sum_{\pi \in \mathcal{WOP}_n(213), \text{mindes}(\pi)=0} x^{\ell(\pi)} = \frac{1 + t(-2 + t - tx)}{1 + t^2 - t(2+x)}.$$

The first few terms of $1 + \sum_{n\geq 1} t^n \sum_{\pi \in \mathcal{WOP}_n(213), \text{mindes}(\pi)=0} x^{\ell(\pi)}$ are

$$\begin{aligned} &1 + tx + (x + x^2)t^2 + (x + 3x^2 + x^3)t^3 + (x + 6x^2 + 5x^3 + x^4)t^4 + \\ &(x + 10x^2 + 15x^3 + 7x^4 + x^5)t^5 + (x + 15x^2 + 35x^3 + 28x^4 + 9x^5 + x^6)t^6 + \\ &(x + 21x^2 + 70x^3 + 84x^4 + 45x^5 + 11x^6 + x^7)t^7 + \\ &(x + 28x^2 + 126x^3 + 210x^4 + 165x^5 + 66x^6 + 13x^7 + x^8)t^8 + \\ &(x + 36x^2 + 210x^3 + 462x^4 + 495x^5 + 286x^6 + 91x^7 + 15x^8 + x^9)t^9 + \cdots. \end{aligned}$$

Seting $x = 1$ in this series gives the sequence

$$1, 1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181, 10946, 28657, \ldots$$

which is sequence A001519 in the OEIS [11] which has a large number of combinatorial interpretations.

Given any sequence of positive numbers $1 \leq b_1 < b_2 < \cdots < b_s$, we let

$$A = A(x,y,t,q_1,\ldots,q_s) = \mathbb{WOP}^{\text{mindes}}_{213,\{b_1,\ldots,b_s\}}(x,y,t,q_1,\ldots,q_s).$$

It follows from the part structure pictured in Figure 3 and our analysis above that

$$\begin{aligned} A &= 1+\sum_{i=1}^{s} xq_i t^{b_i}(1+y(A-1)+\sum_{a=1}^{b_i}(A-1)(1+y(A-1))^{b_i+1-a}) \\ &= 1+\sum_{i=1}^{s} xq_i t^{b_i}(1+y(A-1))\left(1+\frac{(1+y(A-1))^{b_i}-1}{y}\right). \end{aligned}$$

If we set $F = F(x,y,t,q_1,\ldots,q_s) = A(x,y,t,q_1,\ldots,q_s)-1$, then we see that

$$F = x\sum_{i=1}^{s} q_i t^{b_i}(1+yF)\left(1+\frac{(1+yF)^{b_i}-1}{y}\right).$$

It follows from the Lagrange Inversion Theorem that

$$F|_{x^k} = \frac{1}{k}\delta^k(x)|_{x^{k-1}}$$

where $\delta(x) = \sum_{i=1}^{s} q_i t^{b_i}(1+yx)\left(1+\frac{(1+yx)^{b_i}-1}{y}\right)$.

One can use this expression to show that if $\alpha_1,\ldots,\alpha_s$ are non-negative integers such that $\sum_{i=1}^{s} = k$ and $\sum_{i=1}^{s}\alpha_i b_i = n$, then

$$F|_{x^k t^n q_1^{\alpha_1}\cdots q_s^{\alpha_s}} = \frac{1}{k}\binom{k}{\alpha_1,\ldots,\alpha_s}\frac{(1+xy)^k}{y^k}\prod_{i=1}^{s}\left((1+xy)^{b_i}-1\right)^{\alpha_i}|_{x^{k-1}}.$$

Hence it is possible to get a closed expression for $F_{x^k t^n q_1^{\alpha_1}\cdots q_s^{\alpha_s}}$. We shall omit the details since it is messy.

### 4.4 The function $\mathbb{WOP}^{\text{mindes}}_{123,\{1,2\}}(x,y,t,q_1,q_2)$

Next let us consider the computation of the generating function

$$A(x,y,t,q_1,q_2) = \mathbb{WOP}^{\text{mindes}}_{123,\{1,2\}}(x,y,t,q_1,q_2).$$

We will again consider that the case analysis of elements $\pi = B_1/\ldots/B_j \in \mathcal{WOP}_{n,\{1,2\}}(123)$ by the first return of the path $\Psi(w(\pi))$ so we will keep the same notation. That is we shall assume the first return is at $(n-k,k)$, $B_1/\ldots/B_i$ are the parts containing the numbers $\{k+1,\ldots,n\}$ and $B_{i+1}/\ldots/B_j$ are the parts containing the number $\{1,\ldots,k\}$.

**Case 1.** The first return of $P$ is at the point $(1,n-1)$.

In this case, we showed that $B_1 = \{n\}$. is $xtq_1(1 + y(A(x, y, t, q_1, q_2) - 1)$. If $n = 1$, then we get a contribution of $xtq_1$ and otherwise, $n$ will cause a min-descent between $B_1$ and $B_2$ which will give a contribution of $xtq_1y(A(x, y, t, q_1, q_2) - 1)$. Thus, the contribution in this case is

$$A(x, y, t, q_1, q_2).$$

**Case 2.** The first return of $P$ is at the point $(2, n - 2)$.

In this case, we showed that either $B_1 = \{n - 1\}$ and $B_2 = \{n\}$ or $B_1 = \{n - 1, n\}$. It is easy to see that in the first case, the contribution to $A(x, y, t, q_1, q_2)$ is $x^2t^2q_1^2(1 + y(A(x, y, t, q_1, q_2) - 1)$. That is, if $n = 2$, then we get a contribution of $x^2t^2q_1^2$ and otherwise, $B_2$ will cause a min-descent between $B_2$ and $B_3$ which will give a contribution of $x^2t^2q_1^2(y(A(x, y, t, q_1, q_2) - 1)$. Similarly, in the second case the contribution to $A(x, y, t, q_1, q_2)$ is $xt^2q_2(1 + y(A(x, y, t, q_1, q_2) - 1))$ as there will be min-descent between $B_1$ and $B_2$ if $B_2$ exists. Thus the total contribution to $A(x, y, t, q_1, q_2)$ from Case 2 is

$$(x^2t^2q_1^2 + xt^2q_2)(1 + y(A(x, y, t, q_1, q_2) - 1)).$$

**Case 3.** $k < n - 2$ and $k + 1$ is in column $n - k - 1$.
In this case, we have the situation pictured in Figure 9. Thus $w(\pi) = w_1 \dots w_n$ where $w_{n-k-1} = k+1$ and $w_{n-k} = p$ where $k+1 < p$. It follows that either $B_i = \{k+1, p\}$ or $B_{i-1} = \{k+1\}$ and $B_i = \{p\}$. We claim that the contribution to $A(x, y, t, q_1, q_2)$ in the first case where $B_i = \{k + 1, p\}$ is

$$y(A(x, y, t, q_1, q_2) - 1)xt^2q_2(1 + y(A(x, y, t, q_1, q_2) - 1)).$$

That is, the first factor of $y$ comes from the fact that there is a min-descent between $B_{i-1}$ and $B_i$ since the first element of $B_i$ is $k + 1$ which is the smallest element in $B_1/ \dots /B_i$. The next factor $(A(x, y, t, q_1, q_2) - 1)$ comes from summing over the weights of the reductions of $B_1/ \dots /B_{i-1}$ over all possible choices of $B_1/ \dots /B_{i-1}$. The factor $xt^2q_2$ comes from $B_i$. If $B_{i+1}/ \dots /B_j$ is empty then we get a factor of 1 and, if $B_{i+1}/ \dots /B_j$ is not empty, then we get a factor of $y$, coming from the fact that the minimal element of $B_i$ which is $k + 1$ is greater than the minimal element of $B_{i+1}$ which is some element in $\{1, \dots, k\}$, and a factor of $(A(x, y, t, q_1, q_2) - 1)$ coming summing the weights over all possible choices of $B_{i+1}/ \dots /B_j$.

A similar reasoning will show that the contribution to $A(x, y, t, q_1, q_2)$ in the first case where $B_{i-1} = \{k + 1\}$ and $B_i = \{p\}$ is

$$y(A(x, y, t, q_1, q_2) - 1)x^2t^2q_1^2(1 + y(A(x, y, t, q_1, q_2) - 1)).$$

Thus the total contribution to $A(x, y, t, q_1, q_2)$ in Case 3 is

$$y(A(x, y, t, q_1, q_2) - 1)(xt^2q_2 + x^2t^2q_1^2)(1 + y(A(x, y, t, q_1, q_2) - 1)).$$

At this point, our analysis differs from our analysis in $\mathbb{WOP}_{123,\{1,2\}}(x, y, t, q_1, q_2)$.

**Case 4** $k < n - 2$, $k + 1$ is in column $r = n - k - 2$ and $B_{i-1} = \{k + 1, p_1\}$.
Refer to Figure 10, the word $w(\pi) = w_1 \dots w_n$ where $w_{n-k-2} = k+1$ and $w_{n-k-1} = p_1$, $w_{n-k} = p_2$, where $k + 1 < p_2 < p_1$. It follows that $B_i = \{p_2\}$. Since $B_{i-1} = \{k + 1, p_1\}$, there will be no min-descent between $B_{i-1}$ and $B_i$. Refer to the Dyck path structure in Figure 15 that if the path ends with 3 right steps $RRR$ and it does not have a return, then there are two sub-Dyck path

component denoted $B$ in the picture – the part tracking back from last step before the last down step to the step that it first reaches the first diagonal, and the part from the next step back to the start point. The corresponding part in ordered set partition side is parts $B_1, \ldots, B_{i-2}$ that can be seen as 2 ordered set partitions that avoid 123, whose contribute is $(1 + y(A(x, y, t, q_1, q_2) - 1))^2$. The contribution of part $B_{i-1}$ and $B_i$ is $t^3 q_1 q_2$ and the contribution of blocks $B_{i+1}/\ldots/B_j$ is $(1 + y(A(x, y, t, q_1, q_2) - 1))$ for the same reason as Case 3. Thus the contribution of this case is

$$(1 + y(A(x, y, t, q_1, q_2) - 1))^2 x t^3 q_1 q_2 (1 + y(A(x, y, t, q_1, q_2) - 1)).$$

**Case 5**. $k < n - 2$ and $\pi$ does not satisfy Case 4.

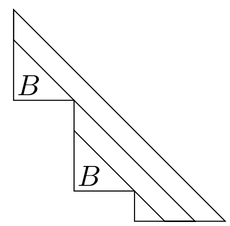


Figure 15: The situation in Case 3.

This case is similar to Case 4 of analysis of $\mathbb{WOP}^{\text{des}}_{123,\{1,2\}}(x, y, t, q_1, q_2)$ in Section 3. In this case, $B_i$ must be a singleton, and we claim that the contribution of this case is

$$y\left(A(x, y, t, q_1, q_2) - 1 - xtq_1(1 + y(A(x, y, t, q_1, q_2) - 1)) - xt^2 q_2(1 + y(A(x, y, t, q_1, q_2) - 1))^2\right) \cdot xtq_1(1 + y(A(x, y, t, q_1, q_2) - 1)).$$

That is, the first factor of $y$ comes from the fact that there is a descent caused by the last element of $B_{i-1}$ and the element in $B_i$. The next factor comes summing over the weights of the reductions of $B_1/\ldots/B_{i-1}$ over all possible satisfying choices of $B_1/\ldots/B_{i-1}$. The contribution of part $B_i$ is $tq_1$ and the last factor $(1 + y(A(x, y, t, q_1, q_2) - 1))$ is the contribution of blocks $B_{i+1}/\ldots/B_j$.

Adding up the contribution lead to the following theorem.

**Theorem 19.** *The function* $\mathbb{WOP}^{\text{mindes}}_{123,\{1,2\}}(x, y, t, q_1, q_2)$ *is the root of the following degree* 3 *polynomial equation about* $A$*,*

$$\begin{aligned} A = 1 &+ txq_1(1 + y(A - 1)) + (t^2 x q_2 + t^2 x^2 q_1^2)(1 + y(A - 1))^2 + t^3 x q_1 q_2 (1 + y(A - 1))^3 \\ &+ txyq_1(1 + y(A - 1))(A - 1 - txq_1(1 + y(A - 1)) - t^2 x q_2 (1 + y(A - 1))^2). \end{aligned}$$

### 4.5 The function $\mathbb{WOP}^{\text{mindes}}_{321}(x, y, t)$

We write $C(x, y, t) = \mathbb{WOP}^{\text{mindes}}_{321}(x, y, t)$. To study the function $C(x, y, t)$, we use the fact that the reverse of the word of any $\pi \in \mathcal{WOP}(321)$ is 123-avoiding. In other word, if we let $\overline{\mathcal{WOP}}(123)$ be the set of ordered set partitions whose numbers are organized in decreasing order inside each part and the word is 123-avoiding, then each $\pi \in \mathcal{WOP}(321)$ is correspond with a $\bar{\pi} \in \overline{\mathcal{WOP}}(123)$. The pdes of $\pi$ is then equal to the rise of the minimal elements of consecutive blocks (or minrise) of $\bar{\pi}$. We want to work on $\overline{\mathcal{WOP}}(123)$ and the statistic minrise to compute the function $C(x, y, t)$.

We also need to define $C_\ell(x, y, t)$ as the generating functions tracking the number of minrise without tracking the minrise caused by the last two parts of ordered set partitions in $\overline{\mathcal{WOP}}(123)$ that

$$C_\ell(x, y, t) = 1 + \sum_{n \geq 1} t^n \sum_{\pi \in \overline{\mathcal{WOP}}_n(123)} x^{\ell(\pi)} y^{|\{i : i < \ell(\pi) - 1, B_i <_{min} B_{i+1}\}|}.$$

We will always use $C$ and $C_\ell$ for short of $C(x,y,t)$ and $C_\ell(x,y,t)$.

We begin with studying the function $C(x,y,t)$. Note that the action *lift* defined in Section 3 keeps the min-descents for any ordered set partitions in $\overline{\mathcal{WOP}}_n(123)$, which makes it possible to do recursions of $\overline{\mathcal{WOP}}_n(123)$ using the Dyck path bijection. For any $\pi = B_1/\cdots/B_m \in \overline{\mathcal{WOP}}_n(123)$, we let $w(\pi) = w_1\cdots w_n \in S_n(123)$. Let the first return of the corresponding Dyck path be at the $n-k^{\text{th}}$ column and let the number $w_{n-k}$ be in the block $B_i$.

Then there are 5 Cases.

**Case 1.** $B_i$ has size 1 and $w_{n-k-1} = k+1$.
In this case, there is a minrise between part $B_{i-1}$ and $B_i$. The numbers before $k+1$ reduce to an ordered set partition in $\overline{\mathcal{WOP}}_{n-k-2}(123)$. There are two possibility that $B_{i-1}$ either only has the number $k+1$ or contains other numbers, and in the later case the minrise caused by last two parts in the previous numbers will not be count. Thus the contribution of numbers before $k+1$ to function $C(x,y,t)$ is $tx(C+\frac{C_\ell-1}{x})$. Since the numbers after $w_{n-k}$ can form any ordered set partitions in $\overline{\mathcal{WOP}}_k(123)$ and the minrise is not affected, we have that the contribution to function $C(x,y,t)$ of this case is

$$t^2x^2y\left(C+\frac{C_\ell-1}{x}\right)C.$$

**Case 2.** $B_i$ has size$>1$ and $w_{n-k-1} = k+1$.
In this case, $B_i$ contains no numbers in $\{w_1,\dots,w_{n-k-1}\}$ and there will not be a minrise between part $B_{i-1}$ and $B_i$. The contribution of the numbers before $w_{n-k}$ is $tx(C+\frac{C_\ell-1}{x})$, and the contribution of the numbers from $w_{n-k}$ is $tx\left(\frac{C-1}{x}\right)$, and the contribution to function $C(x,y,t)$ of this case is

$$t^2x^2\left(C+\frac{C_\ell-1}{x}\right)\left(\frac{C-1}{x}\right).$$

**Case 3.** $B_i$ has size 1 and $w_{n-k-1} \neq k+1$.
In this case, there will not be a minrise between part $B_{i-1}$ and $B_i$. The contribution of the numbers before $w_{n-k}$ is $\left(C - tx\left(C+\frac{C_\ell-1}{x}\right)\right)$. Since the numbers after $w_{n-k}$ forms an ordered set partition in $\overline{\mathcal{WOP}}_k(123)$ and the first part can either joint the number $w_{n-k}$ or not without changing the minrise, thus the contribution of the numbers from $w_{n-k}$ is $tx\left(C+\frac{C-1}{x}\right)$, and the contribution to function $C(x,y,t)$ of this case is

$$tx\left(C+\frac{C-1}{x}\right)\left(C-tx\left(C+\frac{C_\ell-1}{x}\right)\right).$$

**Case 4.** $w_{n-k-1} \in B_i$ but $w_{n-k+1} \notin B_i$.
In this case, there will not be a minrise between part $B_{i-1}$ and $B_i$. We have $w_{n-k} \neq k+1$ and $w_{n-k-1} \neq k+1$ in order to satisfy that $w_{n-k-1} \in B_i$. $w_{n-k+1} \notin B_i$ implies that the first part of the ordered set partition after $w_{n-k}$ does not joint the number $w_{n-k}$. Thus the numbers up tp $w_{n-k}$ contributes $t\left(C-1-tx\left(C+\frac{C_\ell-1}{x}\right)\right)$ and the numbers after $w_{n-k}$ contributes $C$ to the function $C(x,y,t)$. Thus the total contribution of this case is

$$tC\left(C-1-tx\left(C+\frac{C_\ell-1}{x}\right)\right).$$

**Case 5.** $w_{n-k-1} \in B_i$ and $w_{n-k+1} \in B_i$.
In this case, there will still be no minrise between part $B_{i-1}$ and $B_i$. We have $w_{n-k} \neq k+1$ and

$w_{n-k-1} \neq k+1$ in order to satisfy that $w_{n-k-1} \in B_i$. $w_{n-k+1} \in B_i$ implies that the first part of the ordered set partition after $w_{n-k}$ joints the number $w_{n-k}$. As part $B_i$ connects the numbers before $w_{n-k}$ and the numbers after $w_{n-k}$, the mindes caused by the last two parts before $w_{n-k}$ will not be count. Thus the numbers up tp $w_{n-k}$ contributes $t\left(C_\ell - 1 - tx\left(C + \frac{C_\ell - 1}{x}\right)\right)$ and the numbers after $w_{n-k}$ contributes $\frac{C-1}{x}$ to the function $C(x, y, t)$. The total contribution of this case is

$$t\left(\frac{C-1}{x}\right)\left(C_\ell - 1 - tx\left(C + \frac{C_\ell - 1}{x}\right)\right).$$

Summing over all the five cases, we have that

$$\begin{aligned} C(x,y,t) &= 1 + (y-1)t^2x^2C\left(C + \frac{C_\ell - 1}{x}\right) + txC\left(C + \frac{C-1}{x}\right) \\ &\quad + tC\left(C - 1 - tx\left(C + \frac{C_\ell - 1}{x}\right)\right) + t\left(\frac{C-1}{x}\right)\left(C_\ell - 1 - tx\left(C + \frac{C_\ell - 1}{x}\right)\right). \end{aligned}$$

We can do similar analysis for $C_\ell(x, y, t)$. We have the following 7 cases, of which the first 5 cases are similar to that of $C(x, y, t)$.

**Case 1.** $B_i$ has size 1, $w_{n-k-1} = k + 1$ and $k > 0$.
The argument is same as Case 1 of $C(x, y, t)$ except that the contribution of the numbers after $w_{n-k}$ is $C_\ell - 1$ instead of $C$, since $k > 0$ implies that $B_{i+1}$ is not empty, and we are not counting the minrise between the last two parts of $\pi$.Thus the contribution to $C_\ell(x, y, t)$ of this case is

$$t^2x^2y\left(C + \frac{C_\ell - 1}{x}\right)(C_\ell - 1).$$

**Case 2.** $B_i$ has size$>1$ and $w_{n-k-1} = k + 1$.
Similar to Case 2 of $C(x, y, t)$ that the contribution is $t^2x^2\left(C + \frac{C_\ell - 1}{x}\right)\left(\frac{C_\ell - 1}{x}\right)$. The only difference is that the contribution of numbers after $w_{n-k}$ is $\frac{C_\ell - 1}{x}$ instead of $\frac{C-1}{x}$ as we are not counting the minrise of last two parts.

**Case 3.** $B_i$ has size 1, $w_{n-k-1} \neq k + 1$ and $k > 0$.
Similar to Case 3 of $C(x, y, t)$ that the contribution is $tx\left(C_\ell - 1 + \frac{C_\ell - 1}{x}\right)\left(C - tx\left(C + \frac{C_\ell - 1}{x}\right)\right)$. The difference is that the contribution of numbers after $w_{n-k}$ is $\left(C_\ell - 1 + \frac{C_\ell - 1}{x}\right)$ as we are not counting the minrise of last two parts and the collection of numbers after $w_{n-k}$ is not empty.

**Case 4.** $w_{n-k-1} \in B_i$, $w_{n-k+1} \notin B_i$ and $k > 0$.
Similar to Case 4 of $C(x, y, t)$ that the contribution is $t(C_\ell - 1)\left(C - 1 - tx\left(C + \frac{C_\ell - 1}{x}\right)\right)$. The difference is that the contribution of numbers after $w_{n-k}$ is $(C_\ell - 1)$ since $k > 0$ implies that the collection of numbers after $w_{n-k}$ is not empty.

**Case 5.** $w_{n-k-1} \in B_i$ and $w_{n-k+1} \in B_i$.
Similar to Case 5 of $C(x, y, t)$ that the contribution is $t\left(\frac{C_\ell - 1}{x}\right)\left(C_\ell - 1 - tx\left(C + \frac{C_\ell - 1}{x}\right)\right)$. The difference is that the contribution of numbers after $w_{n-k}$ is $\frac{C_\ell - 1}{x}$ as we are not counting the minrise of last two parts.

**Case 6.** $k = 0$ and $w_{n-k-1} \notin B_i$.
In this case, $B_i$ has only the number $w_{n-k}$. Since we are not counting the descents of the last two

parts, we do not care whether $w_{n-k}$ is bigger or smaller than the minimum of the previous last part. The contribution of this case will be $txC$.

**Case 7.** $k=0$ and $w_{n-k-1}\in B_i$.
In this case, $B_i$ can be seen as enlarging the last block before the number $w_{n-k}$ to $w_{n-k}$. The last minrise before $w_{n-k}$ will not be counted, and $w_{n-k}, w_{n-k-1}\neq k+1$. The contribution of this case is $t\left(C_\ell-1-tx\left(C+\frac{C_\ell-1}{x}\right)\right)$.

Summing over all the 7 cases, we have that

$$\begin{aligned}C_\ell(x,y,t) \;=\;& 1+(y-1)t^2x^2(C_\ell-1)\left(C+\frac{C_\ell-1}{x}\right)+txC\left(C_\ell-1+\frac{C_\ell-1}{x}\right)+txC\\&+t\left(C_\ell-1-tx\left(C+\frac{C_\ell-1}{x}\right)\right)+tx\left(C_\ell-1+\frac{C_\ell-1}{x}\right)\left(C-tx\left(C+\frac{C_\ell-1}{x}\right)\right)\\&+t(C_\ell-1)\left(C-1-tx\left(C+\frac{C_\ell-1}{x}\right)\right).\end{aligned}$$

With these equations about $C(x,y,t)$ and $C_\ell(x,y,t)$ computed, we can compute the Groebner basis of the functions to find an equation that $C(x,y,t)$ satisfies, and we have the following theorem.
**Theorem 20.** *The function $\mathbb{WOP}_{321}^{\text{mindes}}(x,y,t)$ is the root of the following degree 4 polynomial equation about $C$,*

$$\begin{aligned}&1+t(-1+t-t^2+x(2+2x-xy))\\&+C\left(-2+t(-3+t(3+x(-4+3x(-2+y)))+x(-5+x(-3+y))+t^2(2+x(3+3x-2xy)))\right)\\&+C^2\left(1+t((3+x)^2-t^3(-3+x^2)+t(3+x)(-1+2x(1+x))\right.\\&\qquad\left.-t^2(10+x(6+x(3+x(4+x(-2+y)(-1+y)-3y)-y))))\right)\\&+C^3t\left(-5-3x+t(-7-x(8+x(3+x)(1+y))+t^2(-6+x(-6+x(-3+5y+x(1+x+y-xy))))\right.\\&\qquad\left.+t(18+x(17+x(6-6y+x(2+x-(4+x)y))))))\right)\\&+C^4t^2(2+x-t(1+x+x^2)+tx^2y)\left(3+2x+t(-3-x(3+x)+x^2(2+x)y)\right)=0.\end{aligned}$$

## 5 Generating functions for p-descents

In this section, we shall study the generating functions $\mathbb{WOP}_\alpha^{\text{pdes}}(x,y,t)$ for certain $\alpha\in S_3$. Based on the analysis in Section 2, we need to study the following 4 kinds of generating functions,

$$\begin{aligned}\mathbb{WOP}_{132}^{\text{pdes}}(x,y,t) &= \mathbb{WOP}_{213}^{\text{pdes}}(x,y,t),\\\mathbb{WOP}_{231}^{\text{pdes}}(x,y,t) &= \mathbb{WOP}_{312}^{\text{pdes}}(x,y,t),\\\mathbb{WOP}_{123}^{\text{pdes}}(x,y,t) &\;,\;\; \mathbb{WOP}_{321}^{\text{pdes}}(x,y,t).\end{aligned}$$

We are able to solve the function $\mathbb{WOP}_{132}^{\text{pdes}}(x,y,t)=\mathbb{WOP}_{213}^{\text{pdes}}(x,y,t)$, and write the functions $\mathbb{WOP}_{231}^{\text{pdes}}(x,y,t)=\mathbb{WOP}_{312}^{\text{pdes}}(x,y,t)$ and $\mathbb{WOP}_{321}^{\text{pdes}}(x,y,t)$ as roots of polynomial equations respectively. We don't have results about the function $\mathbb{WOP}_{123}^{\text{pdes}}(x,y,t)$.

### 5.1 The functions $\mathbb{WOP}_{132}^{\text{pdes}}(x,y,t)=\mathbb{WOP}_{213}^{\text{pdes}}(x,y,t)$

As we observed in Section 2,

$$\mathbb{WOP}_{132}^{\text{pdes}}(x,y,t)=\mathbb{WOP}_{132}^{\text{maxdes}}(x,y,t),$$

Thus we have the following theorem.
**Theorem 21.**

$$\begin{aligned}
\mathbb{WOP}_{132}^{\mathrm{pdes}}(x,y,t) &= \mathbb{WOP}_{213}^{\mathrm{pdes}}(x,y,t) \\
&= \mathbb{WOP}_{132}^{\mathrm{maxdes}}(x,y,t) = \mathbb{WOP}_{213}^{\mathrm{mindes}}(x,y,t) = \mathbb{WOP}_{312}^{\mathrm{mindes}}(x,y,t) \\
&= \frac{P(x,y,t) - \sqrt{(Q(x,y,t)}}{R(x,y,t)},
\end{aligned}$$

*where*

$$\begin{aligned}
P(x,y,t) &= 1 - 2t + t^2 - tx + 2ty - 2t^y + txy + 2t^2xy - 2t^2xy^2, \\
Q(x,y,t) &= 1 - 4t + 6t^2 - 4t^3 + t^4 - 2tx + 4t^2x - 2t^3x + t^2x^2 - 2txy + \\
&\quad 4t^2xy - 2t^3xy - 2t^2x^2y + t^2x^2y^2, \text{ and} \\
R(x,y,t) &= 2(ty - t^2y + txy - t^2xy^2),
\end{aligned}$$

*and*

$$\sum_{\pi \in \mathcal{WOP}_{n,k}(132)} y^{\mathrm{pdes}(\pi)} = \frac{1}{k}\frac{1}{y}\sum_{a=0}^{k}\sum_{v=0}^{n-k}\binom{k}{a}\binom{a+v-1}{v}\binom{k-a+(n-k-v)-1}{n-k-v}\binom{k+a+v}{k-1}(y-1)^{k-a}. \tag{24}$$

## 5.2 The functions $\mathbb{WOP}_{231}^{\mathrm{pdes}}(x,y,t) = \mathbb{WOP}_{312}^{\mathrm{pdes}}(x,y,t)$

We compute the function $\mathbb{WOP}_{312}^{\mathrm{pdes}}(x,y,t)$ as a representative of this equivalent class. We write $D(x,y,t) = \mathbb{WOP}_{312}^{\mathrm{pdes}}(x,y,t)$ for our convenience in the analysis. As this is different from the 132-avoiding case, we will consider a new structure for the set $\mathcal{WOP}(312)$.

Given any ordered set partition $\pi \in B_1/\dots/B_k \in \mathcal{WOP}_n(312)$, if the size $n = 0$, then it contributes 1 to the function $D(x,y,t)$. Otherwise, $\pi$ has at least one part and we suppose the last part $B_k = \{a_1, a_2, \dots, a_r\}$ has $r \geq 1$ numbers. Note that there will not be any number number $a > a_2$ in the previous blocks $B_1, \dots, B_{k-1}$, otherwise the subsequence $(a, a_1, a_2)$ of $w(\pi)$ is a 312-match. Thus, the subsequence $\{a_2, \dots, a_r\}$ must be a consecutive integer sequence.

Now, We divide the numbers in the previous blocks $B_1, \dots, B_{k-1}$ into 2 sets, $A_1 = \{1, \dots, a_1 - 1\}$ be the numbers smaller than $a_1$ and $A_2 = \{a_1 + 1, \dots, a_2 - 1\}$ be the numbers bigger than $a_1$. The numbers in set $A_1$ must appear before the numbers in set $A_2$ as otherwise there will be a 312-match in the word. Thus, an ordered set partition $\pi = B_1/\dots/B_k \in \mathcal{WOP}_n(312)$ has the structure pictured in Figure 16.

We let $A_i(\pi)$ the restriction of $\pi$ in set $A_i$, then each $A_i(\pi)$ is also an ordered set partition in $\mathcal{WOP}(312)$. However, if both $A_i$'s are not empty, then it is possible that the last block of $A_1$ joints the first block of $A_2$. In that case, the pdes caused by the last two blocks in $A_1$ (if there is) and the pdes caused by the first two blocks in $A_2$ (if there is) will not be count in the number of pdes in $\pi$. We let $D_\ell(x,y,t)$, $D_f(x,y,t)$ and $D_{\ell f}(x,y,t)$ be the generating functions tracking the number of pdes without tracking the pdes caused by the last two parts, the first two parts, and both last

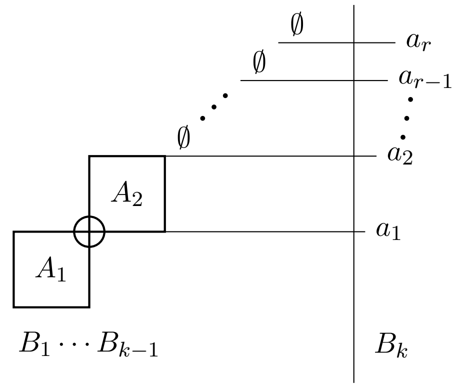


Figure 16: Structure of an ordered set partition in $\mathcal{WOP}(312)$

and first two parts that

$$\begin{aligned}
D_\ell(x,y,t) &= 1+\sum_{n\geq 1} t^n \sum_{\pi\in\mathcal{WOP}_n(312)} x^{\ell(\pi)} y^{|\{i:i<\ell(\pi)-1, B_i >_p B_{i+1}\}|}, \\
D_f(x,y,t) &= 1+\sum_{n\geq 1} t^n \sum_{\pi\in\mathcal{WOP}_n(312)} x^{\ell(\pi)} y^{|\{i:i>1, B_i >_p B_{i+1}\}|}, \\
D_{\ell f}(x,y,t) &= 1+\sum_{n\geq 1} t^n \sum_{\pi\in\mathcal{WOP}_n(312)} x^{\ell(\pi)} y^{|\{i:1<i<\ell(\pi)-1, B_i >_p B_{i+1}\}|},
\end{aligned}$$

then we can compute the recursive equations of functions $D(x,y,t)$, $D_\ell(x,y,t)$, $D_f(x,y,t)$ and $D_{\ell f}(x,y,t)$ respectively.

We first consider the function $D(x,y,t)$.

**Case 1.** The last part $B_k$ has size$>1$.
Then there will always be no pdes involving part $B_k$ as the last part will contain the number $a_2$ greater than any numbers in $B_1,\ldots,B_{k-1}$. The last part has contribution $tx^2+tx^3+\cdots=\frac{tx^2}{1-t}$, and the contribution of $B_1,\ldots,B_{k-1}$ is $D^2(x,y,t)$ when the last block in $A_1$ does not joint the first block of $A_2$, and $\frac{(D_\ell(x,y,t)-1)(D_f(x,y,t)-1)}{x}$ when the last block in $A_1$ joints the first block of $A_2$. Thus, The contribution of this case to the function $D(x,y,t)$ is

$$\frac{tx^2}{1-t}\left(D^2(x,y,t)+\frac{(D_\ell(x,y,t)-1)(D_f(x,y,t)-1)}{x}\right).$$

**Case 2.** $B_k$ has size 1, $A_2$ part only contains 1 block and joints with part $A_1$.
In this case, the set $A_1$ cannot be empty and there will still be no pdes caused by the last two parts of $\pi$. The contribution will be

$$tx\left((D_\ell(x,y,t)-1)\frac{t}{1-t}\right).$$

**Case 3.** $B_k$ has size 1, $A_2$ is empty.
In this case, there will be no pdes caused by the last two parts of $\pi$ and the contribution will be

$$txD(x,y,t).$$

**Case 4.** $B_k$ has size 1, and $\pi$ does not satisfy Case 2 and 3.
In this case, there will be a pdes caused by the last two parts of $\pi$. Considering it is possible that

the last block of $A_1$ joints the first block of $A_2$, we can compute that the contribution of this case is

$$txy\left(D(x,y,t)(D(x,y,t)-1)+\frac{(D_\ell(x,y,t)-1)(D_f(x,y,t)-\frac{tx}{1-t}-1)}{x}\right).$$

Summing over all the 4 cases, and we write $D, D_\ell, D_f, D_{\ell f}$ on the right hand side for short of $D(x,y,t), D_\ell(x,y,t), D_f(x,y,t), D_{\ell f}(x,y,t)$, then we have

$$D(x,y,t)=1+\frac{tx}{1-t}\left(D^2+\frac{(D_\ell-1)(D_f-1)}{x}\right)+(y-1)tx\left(D(D-1)+\frac{(D_\ell-1)(D_f-\frac{tx}{1-t}-1)}{x}\right).$$

Then for the function $D_\ell(x,y,t)$, we do not need to consider the contribution involving part $B_k$, thus the analysis is like Case 1 of function $D(x,y,t)$ and we have

$$D_\ell(x,y,t)=1+\frac{tx}{1-t}\left(D^2+\frac{(D_\ell-1)(D_f-1)}{x}\right).$$

For the function $D_f(x,y,t)$, we have similar cases to function $D(x,y,t)$, but one more case when last part is of size 1.

**Case 1.** The last part $B_k$ has size$>1$.
Then there will always be no pdes involving part $B_k$. The last part has contribution $\frac{tx^2}{1-t}$, and the contribution of $B_1,\ldots,B_{k-1}$ is $(D_f(x,y,t)-1)D(x,y,t)$ when $A_1$ is not empty and the last block in $A_1$ does not joint the first block of $A_2$, and $D_f(x,y,t)$ when $A_1$ is empty, and $\frac{(D_{\ell f}(x,y,t)-1)(D_f(x,y,t)-1)}{x}$ when the last block in $A_1$ joints the first block of $A_2$. Thus, The contribution of this case to the function $D(x,y,t)$ is

$$\frac{tx^2}{1-t}\left((D_f(x,y,t)-1)D(x,y,t)+D_f(x,y,t)+\frac{(D_{\ell f}(x,y,t)-1)(D_f(x,y,t)-1)}{x}\right).$$

**Case 2.** $B_k$ has size 1, $A_2$ part only contains 1 block and joints with part $A_1$.
In this case, the set $A_1$ cannot be empty and there will still be no pdes caused by the last two parts of $\pi$. The contribution will be

$$tx\left((D_{\ell f}(x,y,t)-1)\frac{t}{1-t}\right).$$

**Case 3.** $B_k$ has size 1, $A_2$ is empty.
In this case, there will be no pdes caused by the last two parts of $\pi$ and the contribution will be

$$txD_f(x,y,t).$$

**Case 4.** $B_k$ has size 1, $A_1$ is empty, and $A_2$ only has one block.
In this case, the pdes caused by the only two parts of $\pi$ is not counted as we do not count the first pdes.

$$tx\frac{tx}{1-t}.$$

**Case 5.** $B_k$ has size 1, and the numbers in sets $A_1, A_2$ does not satisfy Case 2, 3 and 4.
In this case, there will be a pdes caused by the last two parts of $\pi$. Considering it is possible that the last block of $A_1$ joints the first block of $A_2$, we can compute that the contribution of this case is

$$txy\left((D_f(x,y,t)-1)(D(x,y,t)-1)+(D_f(x,y,t)-1-\frac{tx}{1-t})+\frac{(D_{\ell f}(x,y,t)-1)(D_f(x,y,t)-\frac{tx}{1-t}-1)}{x}\right).$$

Summing over all the 5 cases, and we write $D, D_\ell, D_f, D_{\ell f}$ on the right hand side for short of $D(x,y,t), D_\ell(x,y,t), D_f(x,y,t), D_{\ell f}(x,y,t)$, then we have

$$D_f(x,y,t) = 1 + \tfrac{tx}{1-t}\left((D_f-1)D + D_f + \tfrac{(D_{\ell f}-1)(D_f-1)}{x}\right) + (y-1)tx\left((D_f-1)D - \tfrac{tx}{1-t} + \tfrac{(D_{\ell f}-1)(D_f-\frac{tx}{1-t}-1)}{x}\right).$$

For the function $D_{\ell f}(x,y,t)$, we do not need to consider the contribution involving part $B_k$, thus the analysis is like Case 1 of function $D_f(x,y,t)$ and we have

$$D_{\ell f}(x,y,t) = 1 + \frac{tx}{1-t}\left((D_f-1)D + D_f + \frac{(D_{\ell f}-1)(D_f-1)}{x}\right).$$

With these equations about $D(x,y,t)$, $D_\ell(x,y,t)$, $D_f(x,y,t)$ and $D_{\ell f}(x,y,t)$ computed, we can compute the Groebner basis of the functions to find an equation that $D(x,y,t)$ satisfies, and we have the following theorem.

**Theorem 22.** *We have the following equations about $D(x,y,t)$, $D_\ell(x,y,t)$, $D_f(x,y,t)$ and $D_{\ell f}(x,y,t)$,*

$$D(x,y,t) = 1 + \tfrac{tx}{1-t}\left(D^2 + \tfrac{(D_\ell-1)(D_f-1)}{x}\right) + (y-1)tx\left(D(D-1) + \tfrac{(D_\ell-1)(D_f-\frac{tx}{1-t}-1)}{x}\right),$$
$$D_\ell(x,y,t) = 1 + \tfrac{tx}{1-t}\left(D^2 + \tfrac{(D_\ell-1)(D_f-1)}{x}\right),$$
$$D_f(x,y,t) = 1 + \tfrac{tx}{1-t}\left((D_f-1)D + D_f + \tfrac{(D_{\ell f}-1)(D_f-1)}{x}\right) + (y-1)tx\left((D_f-1)D - \tfrac{tx}{1-t} + \tfrac{(D_{\ell f}-1)(D_f-\frac{tx}{1-t}-1)}{x}\right),$$
$$D_{\ell f}(x,y,t) = 1 + \tfrac{tx}{1-t}\left((D_f-1)D + D_f + \tfrac{(D_{\ell f}-1)(D_f-1)}{x}\right),$$

*and the function $\mathbb{WOP}_{312}^{\text{pdes}}(x,y,t)$ is the root of the following degree 3 polynomial equation about $D$,*

$$1 - t + D(-1+t)(1+t(1+2x(-1+y))) + D^2(1-t)t\left(1 + tx^2(-1+y)^2 + x(-1+t(-1+y)+2y)\right)$$
$$+ D^3t^2x(-1+y)(-1+t(1+x(-1+y))-xy) = 0.$$

### 5.3 The function $\mathbb{WOP}_{321}^{\text{pdes}}(x,y,t)$

We write $D(x,y,t) = \mathbb{WOP}_{321}^{\text{pdes}}(x,y,t)$. As we defined in Section 4.5, $\overline{\mathcal{WOP}}(123)$ is the set of ordered set partitions whose numbers are organized in decreasing order inside each part and the word is 123-avoiding. Each $\pi \in \mathcal{WOP}(321)$ is correspond with a $\bar{\pi} \in \overline{\mathcal{WOP}}(123)$, and the pdes of $\pi$ is then equal to the part-rise (or prise) of $\bar{\pi}$. We want to work on $\overline{\mathcal{WOP}}(123)$ and the statistic prise to compute the function $D(x,y,t)$.

We also need to define $D_\ell(x,y,t)$, $D_f(x,y,t)$ and $D_{\ell f}(x,y,t)$ as the generating functions tracking the number of prise without tracking the prise caused by the last two parts, the first two parts, and both last and first two parts of ordered set partitions in $\overline{\mathcal{WOP}}(123)$ that

$$\begin{aligned}
D_\ell(x,y,t) &= 1 + \sum_{n\geq 1} t^n \sum_{\pi\in\overline{\mathcal{WOP}}_n(123)} x^{\ell(\pi)} y^{|\{i: i<\ell(\pi)-1, B_i <_p B_{i+1}\}|},\\
D_f(x,y,t) &= 1 + \sum_{n\geq 1} t^n \sum_{\pi\in\overline{\mathcal{WOP}}_n(123)} x^{\ell(\pi)} y^{|\{i: i>1, B_i <_p B_{i+1}\}|},\\
D_{\ell f}(x,y,t) &= 1 + \sum_{n\geq 1} t^n \sum_{\pi\in\overline{\mathcal{WOP}}_n(123)} x^{\ell(\pi)} y^{|\{i: 1<i<\ell(\pi)-1, B_i <_p B_{i+1}\}|}.
\end{aligned}$$

We will always use $D$, $D_\ell$, $D_f$ and $D_{\ell f}$ for short of $D(x,y,t)$, $D_\ell(x,y,t)$, $D_f(x,y,t)$ and $D_{\ell f}(x,y,t)$. As we are generally looking at the same cases as Section 4.5, we shall briefly describe our idea about the recursions of the 4 functions.

For any $\pi = B_1/\cdots/B_m \in \overline{\mathcal{WOP}}_n(123)$, we let $w(\pi) = w_1\cdots w_n \in S_n(123)$. Let the first return of the corresponding Dyck path be at the $n-k^{\text{th}}$ column and let the number $w_{n-k}$ be in the block $B_i$.

For the function $D(x,y,t)$, there are 4 Cases similar to Section 4.5. We will directly give the contribution of each case to function $D(x,y,t)$.

**Case 1.** both $B_{i-1}$ and $B_i$ are of size 1.
The contribution to function $D(x,y,t)$ is $t^2x^2yD^2$.

**Case 2.** $w_{n-k-1} \notin B_i$ but $\pi$ does not satisfy Case 1.
The contribution to function $D(x,y,t)$ is $txD\left(D+\frac{D_f-1}{x}\right) - t^2x^2D^2$.

**Case 3.** $w_{n-k-1} \in B_i$ but $w_{n-k+1} \notin B_i$.
The contribution to function $D(x,y,t)$ is $\left(D-1-xt\left(D+\frac{D_\ell-1}{x}\right)\right)\cdot tD$.

**Case 4.** $w_{n-k-1} \in B_i$ and $w_{n-k+1} \in B_i$.
The contribution to function $D(x,y,t)$ is $\left(D_\ell-1-xt\left(D+\frac{D_\ell-1}{x}\right)\right)\cdot t\frac{D_f-1}{x}$.

Summing over all the 4 cases, we have that

$$\begin{aligned}
D(x,y,t) &= 1+t^2x^2(y-1)D^2+txD\left(D+\frac{D_f-1}{x}\right)\\
&\quad +tD\left(D-1-xt\left(D+\frac{D_\ell-1}{x}\right)\right)\\
&\quad +t\left(\frac{D_f-1}{x}\right)\left(D_\ell-1-xt\left(D+\frac{D_\ell-1}{x}\right)\right).
\end{aligned}$$

For the function $D_\ell(x,y,t)$, there are 6 Cases.

**Case 1.** both $B_{i-1}$ and $B_i$ are of size 1, and $k>0$.
The contribution to function $D_\ell(x,y,t)$ is $t^2x^2yD(D_\ell-1)$.

**Case 2.** $w_{n-k-1} \notin B_i$ and $k>0$, but $\pi$ does not satisfy Case 1.
The contribution to function $D_\ell(x,y,t)$ is $txD\left(D_\ell-1+\frac{D_{\ell f}-1}{x}\right) - t^2x^2D(D_\ell-1)$.

**Case 3.** $w_{n-k-1} \in B_i$ and $k>0$, but $w_{n-k+1} \notin B_i$.
The contribution to function $D_\ell(x,y,t)$ is $\left(D-1-xt\left(D+\frac{D_\ell-1}{x}\right)\right)\cdot t(D_\ell-1)$.

**Case 4.** $w_{n-k-1} \in B_i$ and $w_{n-k+1} \in B_i$.
The contribution to function $D_\ell(x,y,t)$ is $\left(D_\ell-1-xt\left(D+\frac{D_\ell-1}{x}\right)\right)\cdot t\frac{D_{\ell f}-1}{x}$.

**Case 5.** $k=0$ and $w_{n-k-1} \notin B_i$.
The contribution to function $D_\ell(x,y,t)$ is $txD$.

**Case 6.** $k=0$ and $w_{n-k-1} \in B_i$.
The contribution to function $D_\ell(x,y,t)$ is $t\left(D_\ell-1-tx\left(D+\frac{D_\ell-1}{x}\right)\right)$.

Summing over all the 6 cases, we have that

$$\begin{aligned} D_\ell(x,y,t) &= 1+txD+t^2x^2(y-1)D(D_\ell-1)+txD\left(D_\ell-1+\frac{D_{\ell f}-1}{x}\right) \\ &+t(D_\ell-1)\left(D-1-xt\left(D+\frac{D_\ell-1}{x}\right)\right) \\ &+t\left(\frac{D_{\ell f}-1}{x}\right)\left(D_\ell-1-xt\left(D+\frac{D_\ell-1}{x}\right)\right) \\ &+t\left(D_\ell-1-tx\left(D+\frac{D_\ell-1}{x}\right)\right). \end{aligned}$$

The functions $D_f(x,y,t)$ and $D_{\ell f}(x,y,t)$ have exactly the same 4 cases and 6 cases as $D(x,y,t)$ and $D_\ell(x,y,t)$. The main difference on the right hand side expansion is that some $D$ and $D_\ell$ becomes $D_f$ and $D_{\ell f}$. We omit the classification of cases and organize the terms of the expressions of $D_f(x,y,t)$ and $D_{\ell f}(x,y,t)$ in the same way as functions $D(x,y,t)$ and $D_\ell(x,y,t)$, then we have

$$\begin{aligned} D_f(x,y,t) &= 1+t^2x^2(y-1)(D_f-1)D+txD_f\left(D+\frac{D_f-1}{x}\right) \\ &+tD\left(D_f-1-xt\left(D_f+\frac{D_{\ell f}-1}{x}\right)\right) \\ &+t\left(\frac{D_f-1}{x}\right)\left(D_{\ell f}-1-xt\left(D_f+\frac{D_{\ell f}-1}{x}\right)\right), \end{aligned}$$

and

$$\begin{aligned} D_{\ell f}(x,y,t) &= 1+txD_f+t^2x^2(y-1)(D_f-1)(D_\ell-1)+txD_f\left(D_\ell-1+\frac{D_{\ell f}-1}{x}\right) \\ &+t(D_\ell-1)\left(D_f-1-xt\left(D_f+\frac{D_{\ell f}-1}{x}\right)\right) \\ &+t\left(\frac{D_{\ell f}-1}{x}\right)\left(D_{\ell f}-1-xt\left(D_f+\frac{D_{\ell f}-1}{x}\right)\right) \\ &+t\left(D_{\ell f}-1-tx\left(D_f+\frac{D_{\ell f}-1}{x}\right)\right). \end{aligned}$$

With the recursive equations of the four functions computed, one can compute the Groebner basis of the functions to find an equation that $D(x,y,t)$ satisfies, and we have the following theorem.

**Theorem 23.** *The function $\mathbb{WOP}_{321}^{\mathrm{pdes}}(x,y,t)$ is the root of the following degree 6 polynomial equation about D,*

$D((-1+D)x+tD(-1-D^2(1+x)^2+2DD(1+x+x^2(-1+y)D)-2x^2(-1+y)D)+D^3t^5x^5(-1+y)^3+D^2t^4x^3(-1+y)^2(-2+2D(1+x)+x(1+x-xy))+t^2D(1+x+D(-2+x(2(-2+y)+x(4+x(-1+y))(-1+y)))-xD(x^2(-1+y)^2+yD)-D^2(1+x)(-1+x(-2+3x(-1+y)+y))D)+Dt^3x(-1+y)D(1+D^2(1+x)^2+2x(-1+x(-1+y))+DD(-2+x^2(4+3x-2y-3xy)D)D)D)D(1+DD(-2+DD(1+tD(1+x-tD(1+x+x^2D)+D(-1+t)D(1+x+tx^2(-1+y)D)+tx^2yD)D)D)D)=0$

# 6 Results on parking functions

A North-East $n \times n$ Dyck path is a lattice path from $(0,0)$ to $(n,n)$ consisting of east and north steps which stays above the diagonal $y = x$.

We can get an $n \times n$ parking function by labeling the cells east of and adjacent to a north step of a Dyck path with numbers $\{1, \ldots, n\}$ such that the numbers in each column is increasing. On the other hand, we can take a parking function as a combination of a Dyck path and an ordered set partition of same part size composition. Figure 17 is an example of a parking function of size 5.

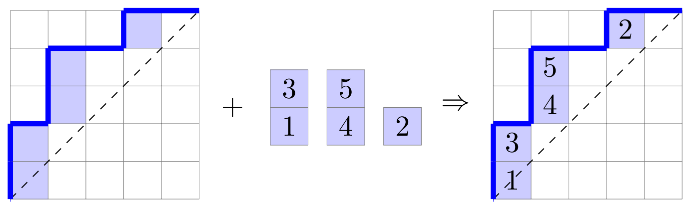


Figure 17: The construction of a parking function.

We say that a parking function avoids a permutation pattern if and only if the ordered set partition of the parking function word-avoids the pattern. We studied pattern avoidance in parking functions in this way and we have the following theorem.

**Theorem 24.** *Let* $\mathrm{pf}_n(123)$ *be the number of* $n \times n$ *parking functions avoid pattern 123 and let* $\mathrm{pf}_{n,k}(123)$ *be the number of* $n \times n$ *parking functions with* $k$ *columns that avoid pattern 123, then*

$$\begin{aligned} \mathrm{pf}_{n,k}(123) &= \frac{1}{(k+1)(n-k+1)} \binom{n}{k} \binom{k}{n-k} \binom{2k}{k} \\ &= \frac{C_k}{n-k+1} \binom{n}{k} \binom{k}{n-k}. \end{aligned}$$

*and*

$$\begin{aligned} \mathrm{pf}_n(123) &= \sum_{k=\frac{n}{2}}^{n} \frac{1}{(k+1)(n-k+1)} \binom{n}{k} \binom{k}{n-k} \binom{2k}{k} \\ &= \sum_{k=\frac{n}{2}}^{n} \frac{C_k}{n-k+1} \binom{n}{k} \binom{k}{n-k}, \end{aligned}$$

*here* $C_k$ *is the* $k^{\text{th}}$ *Catalan number.*

*Proof.* From Theorem 9, we have $\mathrm{wop}_{[b_1,\ldots,b_k]}(123) = C_k$, which implies that for any Dyck path of size $n$ with $k$ columns that are either of size one or of size two, we can find $C_k$ parking functions on the Dyck path. It is not difficult to enumerate that the number of Dyck paths of size $n$ with $k$ columns that are either of size one or of size two is $\frac{\binom{n}{k}\binom{k}{n-k}}{n-k+1}$. Thus

$$\mathrm{pf}_{n,k}(123) = \frac{C_k}{n-k+1} \binom{n}{k} \binom{k}{n-k}.$$

Summing over all possible $\mathrm{pf}_{n,k}(123)$ gives the formula for $\mathrm{pf}_n(123)$. □

# 7 Open problems

In this paper, we mainly use the classical recursion of 132-avoiding permutations and the Dyck path bijection of 123-avoiding permutations to prove results on the generation functions tracking several statistics of ordered set partition that word-avoid some patterns of length 3. Our definition of word-avoidance of an ordered set partition differs from pattern avoidance defined by Godbole, Goyt, Herdan, and Pudwell [4]. Notwithstanding, our definition of 321-word-avoiding ordered set partition coincides $\alpha$-avoiding ordered set partition in the sense of [4] for any pattern $\alpha \in S_3$.

Due to this coincidence, we spent a lot of pages on the problems on the set $\mathcal{WOP}_n(321)$ of ordered set partitions word-avoiding 321. In Section 3, we solve all the generating functions tracking the statistic descents about $\mathcal{WOP}_n(\alpha)$ for any pattern $\alpha$ of length 3, and obtain many beautiful symmetries and multinomial formulas with multinomial coefficients. However, the enumeration for $\mathrm{wop}_{[b_1,\ldots,b_k]}(321) = op_{[b_1,\ldots,b_k]}(321)$ and $\mathrm{wop}_{\langle b_1^{\alpha_1}\ldots b_k^{\alpha_k}\rangle}(321) = op_{\langle b_1^{\alpha_1}\ldots b_k^{\alpha_k}\rangle}(321)$ are still open. As a first question, an explicit formula of $\mathrm{wop}_{\langle b_1^{\alpha_1}\ldots b_k^{\alpha_k}\rangle}(321)$ is desired.

In Section 4 and Section 5, we get nice result about all the generating functions tracking the statistics mindes and pdes, except that we do not have any result about $\mathbb{WOP}_{123}^{\mathrm{pdes}}(x,y,t)$. In particular, we have polynomial equations about the generating functions $\mathbb{WOP}_{321}^{\mathrm{mindes}}(x,y,t)$ and $\mathbb{WOP}_{321}^{\mathrm{pdes}}(x,y,t)$ stated in Section 4.5 and Section 5.3, which still make sense when using pattern avoidance definition in the sense of [4]. The polynomial equations have all the information of the generating functions, and one can come up with efficient recursions easily with the equations. The open problem in this part is the function $\mathbb{WOP}_{123}^{\mathrm{pdes}}(x,y,t)$. We are not able to get recursions about $\mathbb{WOP}_{123}^{\mathrm{pdes}}(x,y,t)$ since the pdes statistic changes abnormally at the action *lift*.

We mentioned pattern avoidance problems about parking functions in Section 6. There is a great number of interesting problems about enumerating parking functions, and all pattern avoiding problems except the formula for $\mathrm{pf}_{n,k}(123)$ are open.

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