# Patterns in Ordered Set Partitions and Parking Functions

Dun Qiu
UC San Diego
duqiu@ucsd.edu
Joint work with Jeffery Remmel

November 16, 2017

# Set Partitions

## Definition (Set Partition)

A set partition $\pi$ of $[n] = \{1, \dots, n\}$ is a family of nonempty, pairwise disjoint subsets $B_1, B_2, \dots, B_k$ of $[n]$ called blocks such that $\bigcup_{i=1}^{k} B_i = [n]$. We write

$$\pi = B_1 / \dots / B_k,$$

where $\min(B_1) < \cdots < \min(B_k)$.

# Set Partitions

## Definition (Set Partition)

A set partition $\pi$ of $[n] = \{1, \ldots, n\}$ is a family of nonempty, pairwise disjoint subsets $B_1, B_2, \ldots, B_k$ of $[n]$ called blocks such that $\bigcup_{i=1}^{k} B_i = [n]$. We write

$$\pi = B_1/\ldots/B_k,$$

where $\min(B_1) < \cdots < \min(B_k)$.

## Example

$\pi = 134/268/57 \vdash [8]$ with parts $B_1 = \{1, 3, 4\}$, $B_2 = \{2, 6, 8\}$, and $B_3 = \{5, 6\}$.

# Ordered Set Partitions

### Definition (Ordered Set Partition)

An ordered set partition with underlying set partition $\pi = B_1/\ldots/B_k$ is a permutation of the blocks of $\pi$, $\delta = B_{\sigma_1}/\ldots/B_{\sigma_k}$ for some permutation $\sigma$ of $[k]$.

# Ordered Set Partitions

## Definition (Ordered Set Partition)

An ordered set partition with underlying set partition $\pi = B_1/\ldots/B_k$ is a permutation of the blocks of $\pi$, $\delta = B_{\sigma_1}/\ldots/B_{\sigma_k}$ for some permutation $\sigma$ of $[k]$.

## Example

$\delta = 57/134/268$ is an ordered set partition of [8] with underlying set partition $\pi = 134/268/57$.

# Ordered Set Partitions

- $OP_n$: the set of order set partitions of $[n]$.

# Ordered Set Partitions

- $OP_n$: the set of order set partitions of $[n]$.
- $OP_{n,k}$: the set of order set partitions of $[n]$ with $k$ parts.

# Ordered Set Partitions

- $OP_n$: the set of order set partitions of $[n]$.
- $OP_{n,k}$: the set of order set partitions of $[n]$ with $k$ parts.
- $OP_{n,k:b_1,\ldots,b_k} = \{B_1/\ldots/B_k \in OP_{n,k} \mid |B_i| = b_i \text{ for } \forall i\}$.

# Ordered Set Partitions

- $OP_n$: the set of order set partitions of $[n]$.
- $OP_{n,k}$: the set of order set partitions of $[n]$ with $k$ parts.
- $OP_{n,k:b_1,\ldots,b_k} = \{B_1/\ldots/B_k \in OP_{n,k} \mid |B_i| = b_i \text{ for } \forall i\}$.

  Let $\lambda = (1^{\alpha_1}2^{\alpha_2}\ldots n^{\alpha_n})$ be a partition and $\ell(\lambda) = \sum_{i=1}^{n} \alpha_i$ denote the length of $\lambda$, then
- $OP_{[\lambda]}$ denote the set of ordered set partitions $\delta = B_1/\ldots/B_{\ell(\lambda)}$ of $|\lambda|$ such that the partition induced by the sizes of the parts of $\delta = \lambda$.

# Ordered Set Partitions

- $OP_n$: the set of order set partitions of $[n]$.
- $OP_{n,k}$: the set of order set partitions of $[n]$ with $k$ parts.
- $OP_{n,k:b_1,\ldots,b_k} = \{B_1/\ldots/B_k \in OP_{n,k} \mid |B_i| = b_i \ for \ \forall i\}$.

  Let $\lambda = (1^{\alpha_1}2^{\alpha_2}\ldots n^{\alpha_n})$ be a partition and $\ell(\lambda) = \sum_{i=1}^{n} \alpha_i$ denote the length of $\lambda$, then
- $OP_{[\lambda]}$ denote the set of ordered set partitions $\delta = B_1/\ldots/B_{\ell(\lambda)}$ of $|\lambda|$ such that the partition induced by the sizes of the parts of $\delta = \lambda$.

Ex. $OP_{[1^1 2^2]} = OP_{5,3:1,2,2} + OP_{5,3:2,1,2} + OP_{5,3:2,2,1}$.

# Reduction of a Sequence

**Definition** ($\mathrm{red}(w)$)

Given a sequence of distinct positive integers $w = w_1 \dots w_n$, we let let the reduction(or standardization) of the sequence $\mathrm{red}(w)$ denote the permutation of $[n]$ obtained from $w$ by replacing the $i$-th smallest letter in $w$ by $i$.

# Reduction of a Sequence

**Definition ($\mathrm{red}(w)$)**

Given a sequence of distinct positive integers $w = w_1 \ldots w_n$, we let let the reduction(or standardization) of the sequence $\mathrm{red}(w)$ denote the permutation of $[n]$ obtained from $w$ by replacing the $i$-th smallest letter in $w$ by $i$.

**Example**

If $w = 4592$, then $\mathrm{red}(w) = 2341$.

# Pattern Avoidance in Ordered Set Partitions

Godbole, Goyt, Herdan, and Pudwell [GGHP, 2014] used the definition that a permutation $\sigma = \sigma_1 \ldots \sigma_j \in \mathcal{S}_j$ *occurs* in an ordered set partition $\delta = B_1/\ldots/B_k$ if and only if there exists $1 \leq i_1 < \cdots < i_j \leq k$ and $b_{i_l} \in B_{i_l}$ for $l = 1, \ldots, j$ such that $\mathrm{red}(b_{i_1} \ldots b_{i_j}) = \sigma$.

$\delta$ *avoids* $\sigma$ if $\sigma$ does not occur in $\delta$.

# Pattern Avoidance in Ordered Set Partitions

Godbole, Goyt, Herdan, and Pudwell [GGHP, 2014] used the definition that a permutation $\sigma = \sigma_1 \dots \sigma_j \in \mathcal{S}_j$ *occurs* in an ordered set partition $\delta = B_1/\dots/B_k$ if and only if there exists $1 \leq i_1 < \cdots < i_j \leq k$ and $b_{i_l} \in B_{i_l}$ for $l = 1, \dots, j$ such that $\mathrm{red}(b_{i_1} \dots b_{i_j}) = \sigma$.

$\delta$ *avoids* $\sigma$ if $\sigma$ does not occur in $\delta$.

Example

$\delta = 57/134/268$, 213 occurs in $\delta$ since $\mathrm{red}(518) = 213$.
But $\delta$ avoids 123 because every element in the first part $\{5, 7\}$ of $\delta$ is bigger than every element in the second part $\{1, 3, 4\}$ of $\delta$.

# Pattern Avoidance in Ordered Set Partitions

If $\sigma = \sigma_1 \ldots \sigma_j$ is a permutation in the symmetric group $\mathcal{S}_j$, then

# Pattern Avoidance in Ordered Set Partitions

If $\sigma = \sigma_1 \dots \sigma_j$ is a permutation in the symmetric group $\mathcal{S}_j$, then

- $OP_n(\sigma) = \{\delta \in OP_n \mid \delta \text{ avoids } \sigma\}$,
- $OP_{n,k}(\sigma) = \{\delta \in OP_{n,k} \mid \delta \text{ avoids } \sigma\}$,
- $OP_{n,k:b_1,\dots,b_k}(\sigma) = \{\delta \in OP_{n,k:b_1,\dots,b_k} \mid \delta \text{ avoids } \sigma\}$,
- $OP_{[\lambda]}(\sigma) = \{\delta \in OP_{[\lambda]} \mid \delta \text{ avoids } \sigma\}$.

# Pattern Avoidance in Ordered Set Partitions

If $\sigma = \sigma_1 \dots \sigma_j$ is a permutation in the symmetric group $\mathcal{S}_j$, then

- $OP_n(\sigma) = \{\delta \in OP_n \mid \delta \text{ avoids } \sigma\}$,
- $OP_{n,k}(\sigma) = \{\delta \in OP_{n,k} \mid \delta \text{ avoids } \sigma\}$,
- $OP_{n,k:b_1,\dots,b_k}(\sigma) = \{\delta \in OP_{n,k:b_1,\dots,b_k} \mid \delta \text{ avoids } \sigma\}$,
- $OP_{[\lambda]}(\sigma) = \{\delta \in OP_{[\lambda]} \mid \delta \text{ avoids } \sigma\}$.

We let

- $op_n(\sigma) = |OP_n(\sigma)|$,
- $op_{n,k}(\sigma) = |OP_{n,k}(\sigma)|$,
- $op_{n,k:b_1,\dots,b_k}(\sigma) = |OP_{n,k;b_1,\dots,b_k}(\sigma)|$, and
- $op_{[\lambda]}(\sigma) = |OP_{[\lambda]}(\sigma)|$.

# The Word of an Ordered Set Partition

Definition ($w(\delta)$)

The word of $\delta = B_{\sigma_1}/\ldots/B_{\sigma_k}$ is obtained from $\delta$ by removing all the slashes, write $w(\delta)$.

# The Word of an Ordered Set Partition

Definition ($w(\delta)$)

The word of $\delta = B_{\sigma_1}/\ldots/B_{\sigma_k}$ is obtained from $\delta$ by removing all the slashes, write $w(\delta)$.

Example

If $\delta = 57/134/268$, then $w(\delta) = 57134268$.

# Focus 1: Word Avoidance in Ordered Set Partitions

## New Definition of Pattern Avoidance

We study an alternative notion of avoidance of permutations in ordered set partitions.

# Focus 1: Word Avoidance in Ordered Set Partitions

## New Definition of Pattern Avoidance

We study an alternative notion of avoidance of permutations in ordered set partitions.

Given an ordered set partition $\delta = B_1/\ldots/B_k$ of $[n]$, we say that a permutation $\sigma = \sigma_1 \ldots \sigma_j$ in the symmetric group $\mathcal{S}_j$ *word occurs* in $\delta$ if there exists $1 \le i_1 < \cdots < i_j \le n$ such that $\mathrm{red}(w_{i_1} \ldots w_{i_j}) = \sigma$ where $w(\delta) = w_1 \ldots w_n$.

# Focus 1: Word Avoidance in Ordered Set Partitions

## New Definition of Pattern Avoidance

We study an alternative notion of avoidance of permutations in ordered set partitions.

Given an ordered set partition $\delta = B_1/\dots/B_k$ of $[n]$, we say that a permutation $\sigma = \sigma_1 \dots \sigma_j$ in the symmetric group $\mathcal{S}_j$ *word occurs* in $\delta$ if there exists $1 \leq i_1 < \cdots < i_j \leq n$ such that $\mathrm{red}(w_{i_1} \dots w_{i_j}) = \sigma$ where $w(\delta) = w_1 \dots w_n$.

Thus $\sigma$ word occurs in $\delta$ if $\sigma$ classically occurs in $w(\delta)$.

We say that an ordered set partition $\delta$ *word avoids* $\sigma$ if $\sigma$ does not word occur in $\delta$.

# Two Kinds of Pattern Occurrences

Example

- Ordered set partition,

# Two Kinds of Pattern Occurrences

Example

- Ordered set partition, $\begin{array}{ccc} 3 & 5 & \\ 1 & 4 & 2 \end{array}$ of $[5] = \{1, 2, 3, 4, 5\}$.

# Two Kinds of Pattern Occurrences

Example

- Ordered set partition, $\begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline \end{array}\ \begin{array}{|c|}\hline 5 \\ \hline 4 \\ \hline \end{array}\ \begin{array}{|c|}\hline 2 \\ \hline \end{array}$ of $[5] = \{1, 2, 3, 4, 5\}$.
- pattern 132, pattern 123

# Two Kinds of Pattern Occurrences

## Example

- Ordered set partition, $\begin{array}{|c|}\hline 3\\\hline 1\\\hline\end{array}\ \begin{array}{|c|}\hline 5\\\hline 4\\\hline\end{array}\ \begin{array}{|c|}\hline 2\\\hline\end{array}$ of $[5] = \{1, 2, 3, 4, 5\}$.
- pattern 132, pattern 123
- $\begin{array}{|c|}\hline 3\\\hline 1\\\hline\end{array}\ \begin{array}{|c|}\hline 5\\\hline 4\\\hline\end{array}\ \begin{array}{|c|}\hline 2\\\hline\end{array}$ and $\begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 5 & 2\\\hline\end{array}$

# Two Kinds of Pattern Occurrences

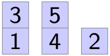
Example

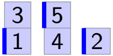
- Ordered set partition, $\begin{smallmatrix}3\\1\end{smallmatrix}\ \begin{smallmatrix}5\\4\end{smallmatrix}\ 2$ of $[5] = \{1, 2, 3, 4, 5\}$.
- pattern 132, pattern 123
- $\begin{smallmatrix}3\\1\end{smallmatrix}\ \begin{smallmatrix}5\\4\end{smallmatrix}\ 2$ and 1 3 4 5 2

  occurrence — word occurrence

# Two Kinds of Pattern Occurrences

**Example**

- Ordered set partition, 

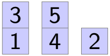
 of $[5] = \{1, 2, 3, 4, 5\}$.
- pattern 132, pattern 123
- 

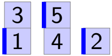
 and 


  occurrence    word occurrence

# Two Kinds of Pattern Occurrences

## Example

- Ordered set partition, 

of $[5] = \{1, 2, 3, 4, 5\}$.
- pattern 132, pattern 123
- 

and 


occurrence word occurrence

# Word Avoidance in Ordered Set Partitions

If $\sigma = \sigma_1 \dots \sigma_j$ is a permutation in the symmetric group $\mathcal{S}_j$, then

# Word Avoidance in Ordered Set Partitions

If $\sigma = \sigma_1 \ldots \sigma_j$ is a permutation in the symmetric group $\mathcal{S}_j$, then

- $WOP_n(\sigma) = \{\delta \in OP_n \mid w(\delta) \text{ avoids } \sigma\}$,
- $WOP_{n,k}(\sigma) = \{\delta \in OP_{n,k} \mid w(\delta) \text{ avoids } \sigma\}$,
- $WOP_{n,k:b_1,\ldots,b_k}(\sigma) = \{\delta \in OP_{n,k:b_1,\ldots,b_k} \mid w(\delta) \text{ avoids } \sigma\}$,
- $WOP_{[\lambda]}(\sigma) = \{\delta \in OP_{[\lambda]} \mid w(\delta) \text{ avoids } \sigma\}$.

# Word Avoidance in Ordered Set Partitions

If $\sigma = \sigma_1 \ldots \sigma_j$ is a permutation in the symmetric group $\mathcal{S}_j$, then

- $WOP_n(\sigma) = \{\delta \in OP_n \mid w(\delta) \text{ avoids } \sigma\}$,
- $WOP_{n,k}(\sigma) = \{\delta \in OP_{n,k} \mid w(\delta) \text{ avoids } \sigma\}$,
- $WOP_{n,k:b_1,\ldots,b_k}(\sigma) = \{\delta \in OP_{n,k:b_1,\ldots,b_k} \mid w(\delta) \text{ avoids } \sigma\}$,
- $WOP_{[\lambda]}(\sigma) = \{\delta \in OP_{[\lambda]} \mid w(\delta) \text{ avoids } \sigma\}$.

Similarly, we let

- $wop_n(\sigma) = |WOP_n(\sigma)|$,
- $wop_{n,k}(\sigma) = |WOP_{n,k}(\sigma)|$,
- $wop_{n,k:b_1,\ldots,b_k}(\sigma) = |WOP_{n,k;b_1,\ldots,b_k}(\sigma)|$, and
- $wop_{[\lambda]}(\sigma) = |WOP_{[\lambda]}(\sigma)|$.

# Word Avoidance v.s. Avoidance

Word avoidance in ordered set partitions is something in-between classical avoidance in permutations and pattern avoidance in ordered set partitions in the sense of Godbole, Goyt, Herdan, and Pudwell [GGHP, 2014].

# Word Avoidance v.s. Avoidance

Word avoidance in ordered set partitions is something in-between classical avoidance in permutations and pattern avoidance in ordered set partitions in the sense of Godbole, Goyt, Herdan, and Pudwell [GGHP, 2014].

Example

There are four ordered set partitions of 3 in which 123 word occurs:

$$123, \ 1/23, \ 12/3 \text{ and } 1/2/3,$$

but 123 occurs in only one permutation of 3, namely 123,
and it occurs in only one ordered partition, of [3], namely $1/2/3$.

# Word Avoidance v.s. Avoidance—Similarity

If $\sigma$ is the decreasing permutation $\sigma = j(j-1)\ldots 21$, then $OP_{n,k;b_1,\ldots,b_k}(\sigma) = WOP_{n,k;b_1,\ldots,b_k}(\sigma)$ for all $n$, $k$, and $b_1, \ldots, b_k$.

# Word Avoidance v.s. Avoidance—Similarity

If $\sigma$ is the decreasing permutation $\sigma = j(j-1)\dots 21$, then $OP_{n,k;b_1,\dots,b_k}(\sigma) = WOP_{n,k;b_1,\dots,b_k}(\sigma)$ for all $n$, $k$, and $b_1, \dots, b_k$.

For example, in [GGHP, 2014], the authors proved that

$$op_{n,k}(12) = op_{n,k}(21) = \binom{n-1}{k-1}.$$

# Word Avoidance v.s. Avoidance—Similarity

If $\sigma$ is the decreasing permutation $\sigma = j(j-1)\dots21$, then $OP_{n,k;b_1,\dots,b_k}(\sigma) = WOP_{n,k;b_1,\dots,b_k}(\sigma)$ for all $n$, $k$, and $b_1,\dots,b_k$.

For example, in [GGHP, 2014], the authors proved that

$$op_{n,k}(12) = op_{n,k}(21) = \binom{n-1}{k-1}.$$

Thus $wop_{n,k}(21) = op_{n,k}(21) = \binom{n-1}{k-1}$.

However, $wop_{n,k}(12) = 1$ since it is easy to see that the only ordered set partition of $[n]$ which word avoids 12 is $n/(n-1)/\dots/2/1$.

# Word Avoidance v.s. Avoidance—Similarity

If $\sigma$ is the decreasing permutation $\sigma = j(j-1)\dots 21$, then $OP_{n,k;b_1,\dots,b_k}(\sigma) = WOP_{n,k;b_1,\dots,b_k}(\sigma)$ for all $n$, $k$, and $b_1,\dots,b_k$.

For example, in [GGHP, 2014], the authors proved that

$$op_{n,k}(12) = op_{n,k}(21) = \binom{n-1}{k-1}.$$

Thus $wop_{n,k}(21) = op_{n,k}(21) = \binom{n-1}{k-1}$.

However, $wop_{n,k}(12) = 1$ since it is easy to see that the only ordered set partition of $[n]$ which word avoids 12 is $n/(n-1)/\dots/2/1$.

For another example, $wop_{n,k}(321) = op_{n,k}(321)$.

# Results on ordered set partitions which word avoid $\sigma$

We have several results about ordered set partitions which word avoid permutations in $\mathcal{S}_3$.

# Results on ordered set partitions which word avoid $\sigma$

We have several results about ordered set partitions which word avoid permutations in $\mathcal{S}_3$.

The enumerations we are interested in are $wop_{n,k}(\sigma)$ and $wop_{[b_1^{\alpha_1},\ldots b_k^{\alpha_k}]}(\sigma)$, which have generating functions

$$A_\sigma(x,t) = \sum_{n\geq 0}\sum_{k\geq 0} wop_{n,k}(\sigma)x^n t^k \quad (1)$$

and

$$A_{\sigma,[b_1,\ldots,b_k]}(x,t,q_1,\ldots,q_k) = \sum_{\alpha_1\geq 0,\ldots,\alpha_k\geq 0} wop_{[b_1^{\alpha_1},\ldots b_k^{\alpha_k}]}(\sigma)x^{\sum_{i=1}^k b_i\alpha_i} t^{\sum_{i=1}^k \alpha_i} q_1^{\alpha_1}\cdots q_k^{\alpha_k}. \quad (2)$$

# The Patterns 132, 213, 231, 312

## Theorem (Pattern 132)

$$A_{132}(x,t) = \frac{x + 1 - \sqrt{x^2 - 4tx - 2x + 1}}{2(1+t)x},$$

# The Patterns 132, 213, 231, 312

## Theorem (Pattern 132)

$$A_{132}(x,t) = \frac{x+1-\sqrt{x^2-4tx-2x+1}}{2(1+t)x},$$

$$wop_{n,k}(132) = \sum_{i=0}^{n/2} \sum_{j=0}^{k-i-1} 2^{k-i-j-1} \binom{n}{j, i-j, i+1, k-i-j-1, n-k-i+j}$$

# The Patterns 132, 213, 231, 312

## Theorem (Pattern 132)

$$A_{132}(x,t) = \frac{x+1-\sqrt{x^2-4tx-2x+1}}{2(1+t)x},$$

$$wop_{n,k}(132) = \sum_{i=0}^{n/2} \sum_{j=0}^{k-i-1} 2^{k-i-j-1} \binom{n}{j, i-j, i+1, k-i-j-1, n-k-i+j}$$

$$wop_{[b_1^{\alpha_1} \dots b_s^{\alpha_s}]}(132) = \frac{1}{n} \binom{k}{\alpha_1 \dots \alpha_s} \binom{n+k}{n-1}.$$

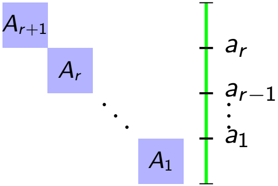
# The Patterns 132, 213, 231, 312

## Theorem (Pattern 132)

$$A_{132}(x,t) = \frac{x+1-\sqrt{x^2-4tx-2x+1}}{2(1+t)x},$$

$$wop_{n,k}(132) = \sum_{i=0}^{n/2}\sum_{j=0}^{k-i-1} 2^{k-i-j-1}\binom{n}{j,\, i-j,\, i+1,\, k-i-j-1,\, n-k-i+j}$$

$$wop_{[b_1^{\alpha_1}\dots b_s^{\alpha_s}]}(132) = \frac{1}{n}\binom{k}{\alpha_1\dots\alpha_s}\binom{n+k}{n-1}.$$




Structure of 132 avoiding ordered set partitions

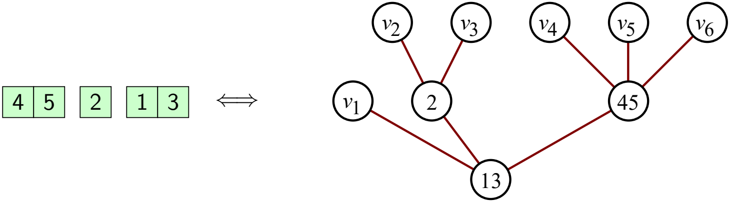
# Bijection between $WOP_n(132)$ and rooted planar trees with no vertices out degree 1

It follows from the Theorem that $wop_n(132)$ is the number of rooted planar trees with $n+1$ leaves that have no vertices of out degree 1.

# Bijection between $WOP_n(132)$ and rooted planar trees with no vertices out degree 1

It follows from the Theorem that $wop_n(132)$ is the number of rooted planar trees with $n+1$ leaves that have no vertices of out degree 1.

In fact, we can give a bijective proof of this.




Bijection between $WOP_n(132)$ and rooted planar trees with no vertices out degree 1

# Symmetry among the Patterns 132,213,231,312

This also allows us to compute $A_{231}(x,t)$, $A_{213}(x,t)$, $A_{312}(x,t)$ since we have produced bijections which prove the following theorem.

# Symmetry among the Patterns 132,213,231,312

This also allows us to compute $A_{231}(x,t)$, $A_{213}(x,t)$, $A_{312}(x,t)$ since we have produced bijections which prove the following theorem.

### Theorem

*For all $n$, $k$, and $b_1, \ldots, b_k$ such that $b_1 + \cdots + b_k = n$,*

$$wop_{n,k}(132) = wop_{n,k}(213) = wop_{n,k}(231) = wop_{n,k}(312) \text{ and}$$

$$\begin{aligned} wop_{n,k;b_1,\ldots,b_k}(132) = wop_{n,k;b_k,\ldots,b_1}(213) &= wop_{n,k;b_1,\ldots,b_k}(231) \\ &= wop_{n,k;b_k,\ldots,b_1}(312). \end{aligned}$$

# Symmetry among the Patterns 132,213,231,312

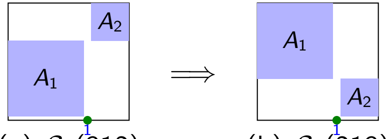
A bijection between $\mathcal{OP}_n(word, 312)$ and $\mathcal{OP}_n(word, 213)$ preserving block size composition

$$wop_{n,k}(132) = \underline{wop_{n,k}(213)} = wop_{n,k}(231) = \underline{wop_{n,k}(312)}$$

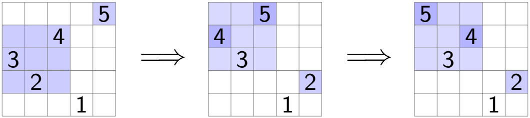
# Symmetry among the Patterns 132,213,231,312

## A bijection between $\mathcal{OP}_n(word, 312)$ and $\mathcal{OP}_n(word, 213)$ preserving block size composition

$$wop_{n,k}(132) = \underline{wop_{n,k}(213)} = wop_{n,k}(231) = \underline{wop_{n,k}(312)}$$




(a) $\mathcal{S}_n(312)$ (b) $\mathcal{S}_n(213)$




$$p = \{3, 2, 4, 1, 5\} \in \mathcal{S}_n(312) \Rightarrow p' = \{5, 3, 4, 1, 2\} \in \mathcal{S}_n(213)$$

# Symmetry between $wop_{n,k;b_1,\ldots,b_k}(321)$ and $wop_{n,k;b_1,\ldots,b_k}(123)$

In [GGHP, 2014], the authors proved that

$$wop_{n,k;b_1,\ldots,b_i,b_{i+1},\ldots,b_k}(321) = wop_{n,k;b_1,\ldots,b_{i+1},b_i,\ldots,b_k}(321).$$

Thus for the permutation 321, we can essentially reduces ourselves to ordered set partitions where the size of the parts weakly increase as we read from left to right. We have proved a similar result for 123.

# Symmetry between $wop_{n,k;b_1,\ldots,b_k}(321)$ and $wop_{n,k;b_1,\ldots,b_k}(123)$

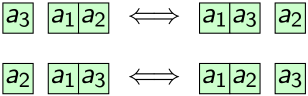
In [GGHP, 2014], the authors proved that

$$wop_{n,k;b_1,\ldots,b_i,b_{i+1},\ldots,b_k}(321) = wop_{n,k;b_1,\ldots,b_{i+1},b_i,\ldots,b_k}(321).$$

Thus for the permutation 321, we can essentially reduces ourselves to ordered set partitions where the size of the parts weakly increase as we read from left to right. We have proved a similar result for 123.

## Theorem

$$wop_{n,k;b_1,\ldots,b_i,b_{i+1},\ldots,b_k}(123) = wop_{n,k;b_1,\ldots,b_{i+1},b_i,\ldots,b_k}(123).$$




$[\ldots,1,2,\ldots]$ $[\ldots,2,1,\ldots]$

# Pattern 123

It is easy to see that an ordered set partition that word avoids 123 can have parts of size only 1 or 2. Then we have the following theorem.

**Theorem**

$$A_{123,[1,2]}(x,t,q_1,q_2) = \frac{1-\sqrt{1-4xt(q_1+xq_2)}}{2tx(q_1+xq_2)}.$$

*And for $2k \geq n$,*

$$wop_{n,k;1^{2k-n},2^{n-k}}(123) = \frac{wop_{n,k}(123)}{\binom{k}{n-k}} = \frac{1}{k+1}\binom{2k}{k} = C_k.$$

*Here $C_k$ is the $k^{\text{th}}$ Catalan number.*

# Pattern 321

We also have generating function of $wop_{n,k}(321)$.

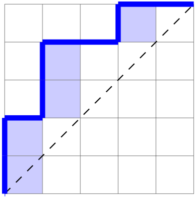
**Theorem**

$$A_{321}(x,t) = \frac{2(t+1)(x-x^2) + t - t\sqrt{1-4(t+1)(x-x^2)}}{2(t+1)^2(x-x^2)}.$$

# Focus 2: Pattern Avoidance in Parking Functions

A parking function of size $n$ can be considered as a combination of a Dyck path on an $n \times n$ lattice and an ordered set partition.

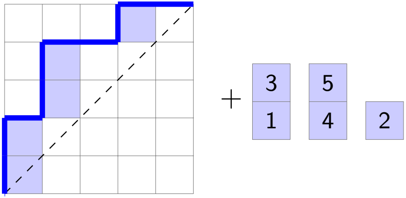
Given any Dyck path on the $n \times n$ lattice, one creates a parking function $P$ by labeling the north steps with $1, 2, \ldots, n$ in such a way that in any column, the numbers are increasing when read from bottom to top.



# Focus 2: Pattern Avoidance in Parking Functions

A parking function of size $n$ can be considered as a combination of a Dyck path on an $n \times n$ lattice and an ordered set partition.

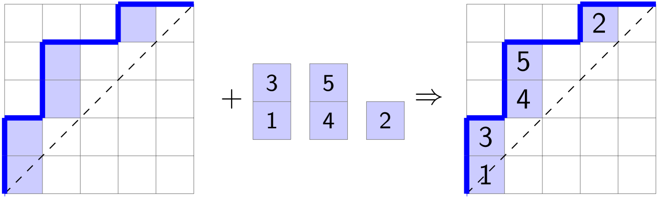
Given any Dyck path on the $n \times n$ lattice, one creates a parking function $P$ by labeling the north steps with $1, 2, \ldots, n$ in such a way that in any column, the numbers are increasing when read from bottom to top.



# Focus 2: Pattern Avoidance in Parking Functions

A parking function of size $n$ can be considered as a combination of a Dyck path on an $n \times n$ lattice and an ordered set partition.

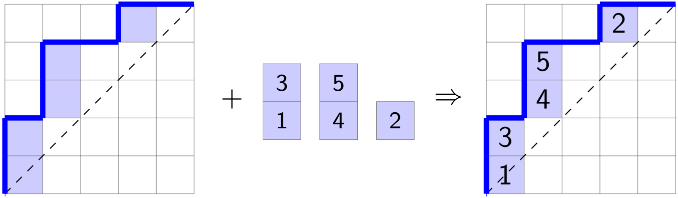
Given any Dyck path on the $n \times n$ lattice, one creates a parking function $P$ by labeling the north steps with $1, 2, \ldots, n$ in such a way that in any column, the numbers are increasing when read from bottom to top.




The construction of a parking function

# Word Avoidance in Parking Functions

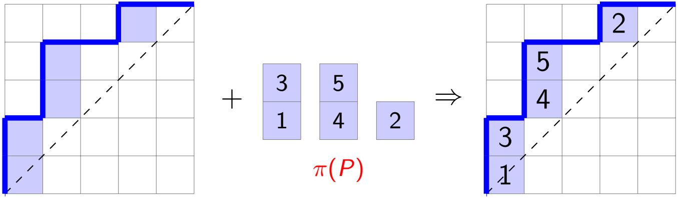
The underlying ordered set partition $\pi(P)$ is the ordered set partition whose parts are the labels of the vertical segments when read from left to right.




The construction of a parking function

# Word Avoidance in Parking Functions

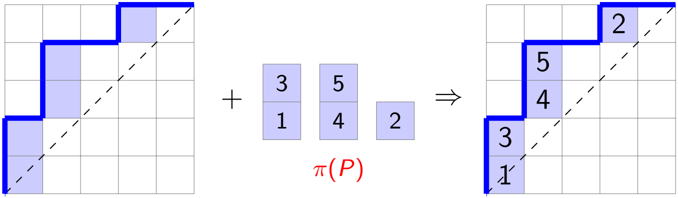
The underlying ordered set partition $\pi(P)$ is the ordered set partition whose parts are the labels of the vertical segments when read from left to right.




The construction of a parking function

# Word Avoidance in Parking Functions

The underlying ordered set partition $\pi(P)$ is the ordered set partition whose parts are the labels of the vertical segments when read from left to right.




The construction of a parking function

We say that a permutation $\sigma$ *word occurs* in a parking function $P$ if it word occurs in $\pi(P)$. A parking function $P$ *word avoids* a permutation $\sigma$ if $\sigma$ does not occur in $P$.

# Word Avoidance in Parking Functions

$PF_n$ is the set of parking functions on the $n \times n$ lattice. If $\sigma = \sigma_1 \dots \sigma_j$ is a permutation in the symmetric group $\mathcal{S}_j$, then

- $PF_n(\sigma) = \{P \in PF_n \mid \pi(P) \in WOP_n\}$,
- $PF_{n,k}(\sigma) = \{P \in PF_n \mid \pi(P) \in WOP_{n,k}\}$,
- $PF_{n,k:b_1,\dots,b_k}(\sigma) = \{P \in PF_n \mid \pi(P) \in WOP_{n,k;b_1,\dots,b_k}\}$,

# Word Avoidance in Parking Functions

$PF_n$ is the set of parking functions on the $n \times n$ lattice. If $\sigma = \sigma_1 \dots \sigma_j$ is a permutation in the symmetric group $\mathcal{S}_j$, then

- $PF_n(\sigma) = \{P \in PF_n \mid \pi(P) \in WOP_n\}$,
- $PF_{n,k}(\sigma) = \{P \in PF_n \mid \pi(P) \in WOP_{n,k}\}$,
- $PF_{n,k:b_1,\dots,b_k}(\sigma) = \{P \in PF_n \mid \pi(P) \in WOP_{n,k;b_1,\dots,b_k}\}$,

Similarly, we let

- $\mathrm{pf}_n(\sigma) = |PF_n(\sigma)|$,
- $\mathrm{pf}_{n,k}(\sigma) = |PF_{n,k}(\sigma)|$,
- $\mathrm{pf}_{n,k:b_1,\dots,b_k}(\sigma) = |PF_{n,k;b_1,\dots,b_k}(\sigma)|$.

# Results on Pattern Avoidance in Parking Functions

We have proved a number of results on patterns in parking functions. For example, we have the following theorem.

**Theorem**

$$\begin{aligned} \mathrm{pf}_{n,k}(123) &= \frac{1}{(k+1)(n-k+1)}\binom{n}{k}\binom{k}{n-k}\binom{2k}{k} \\ &= \frac{C_k}{n-k+1}\binom{n}{k}\binom{k}{n-k}. \end{aligned}$$

*and*

$$\begin{aligned} \mathrm{pf}_{n}(123) &= \sum_{k=\frac{n}{2}}^{n} \frac{1}{(k+1)(n-k+1)}\binom{n}{k}\binom{k}{n-k}\binom{2k}{k} \\ &= \sum_{k=\frac{n}{2}}^{n} \frac{C_k}{n-k+1}\binom{n}{k}\binom{k}{n-k}. \end{aligned}$$

# Focus 1': Adding descents to generating functions

We have also found expression for generating functions over ordered set partitions which word avoid a permutation in $\mathcal{S}_3$ where we keep track of certain kinds of descents.

# Focus 1': Adding descents to generating functions

We have also found expression for generating functions over ordered set partitions which word avoid a permutation in $\mathcal{S}_3$ where we keep track of certain kinds of descents.

Here are three natural types of descent sets that one can define on a ordered set partition $\pi = B_1/\ldots/B_k$. Let $b_{i_{min}} = min\{B_i\}$ and $b_{i_{max}} = max\{B_i\}$.

(a) $Des_{min}(\pi) = \{i \mid b_{i_{min}} > b_{i+1_{min}}\}$,

(b) $Des(\pi) = \{i \mid b_{i_{max}} > b_{i+1_{min}}\}$, and

(c) $\widetilde{Des}(\pi) = \{i \mid b_{i_{min}} > b_{i+1_{max}}\}$.

# Focus 1': Adding descents to generating functions

We have also found expression for generating functions over ordered set partitions which word avoid a permutation in $\mathcal{S}_3$ where we keep track of certain kinds of descents.

Here are three natural types of descent sets that one can define on a ordered set partition $\pi = B_1/\ldots/B_k$. Let $b_{i_{min}} = min\{B_i\}$ and $b_{i_{max}} = max\{B_i\}$.

(a) $Des_{min}(\pi) = \{i \mid b_{i_{min}} > b_{i+1_{min}}\}$,

(b) $Des(\pi) = \{i \mid b_{i_{max}} > b_{i+1_{min}}\}$, and

(c) $\widetilde{Des}(\pi) = \{i \mid b_{i_{min}} > b_{i+1_{max}}\}$.

We let $\mathrm{des}_{min}(\pi) = |Des_{min}(\pi)|$, $\mathrm{des}(\pi) = |Des(\pi)|$, and $\widetilde{\mathrm{des}}(\pi) = |\widetilde{Des}(\pi)|$.

# Focus 1': Adding descents to generating functions

We can find the generating function $\mathrm{des}(\pi)$ and $\widetilde{\mathrm{des}}(\pi)$ over all ordered set partitions which word avoid $\alpha$ for any $\alpha \in \mathcal{S}_3$, and the generating function of $\mathrm{des}_{min}(\pi)$ over all ordered set partitions which word avoid 132.

**Example**

Taking pattern $\sigma = 123$ and $\mathrm{des}(\pi)$, we have

$$A_{\sigma,des}(x,y,t) = \sum_{n\geq 0}\sum_{k\geq 0}\sum_{\pi\in WOP_{n,k}(\sigma)} x^n t^k y^{\mathrm{des}(\pi)}$$

$$= \frac{2t^3x^2(y-1)^2y-2t^2x(2x+1)(y-1)y+t\left(2x^2y+2xy-1\right)-1+(1+t)\sqrt{4t^2x^2y^2-4t^2x^2y-4tx^2y-4txy+1}}{2t(t+1)xy^2(txy-tx-x-1)}.$$

# Future Work

- $wop_{n,k;b_1,\ldots,b_k}(321)$ is not known yet.
- Adding $\mathrm{des}_{min}(\pi)$ to generating function is open for several patterns.
- Many parking function pattern avoidance problems are open.

## References


A. Godbole, A. Goyt, J. Herndan, and L. Pudwell(2014)
Pattern avoidance in ordered set partitions
*Annals of Combinatorics*, **18.3** (2014), 429-445.

# Thank You!

# Thank You!

## Definition (parking function)

Let $\alpha = (a_1, \dots, a_n) \in \mathbb{P}^n$, and let $b_1 \leq b_2 \leq \cdots \leq b_n$ be the increasing rearrangement of $\alpha$. Then $\alpha$ is a parking function *iff* $b_i \leq i$.