Counting classical patterns in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$

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Based on joint work with Jeffrey Remmel

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Outline

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Pattern distribution in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$

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Motivation

Ran Pan's Project P

http://www.math.ucsd.edu/∼projectp/

Problem 13: enumerate permutations in $\mathcal{S}_n$ avoiding a classical pattern and a consecutive pattern at the same time.

Pan, Remmel and I worked on the distribution of consecutive patterns in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$.

Remmel and I started work on the distribution of classical patterns in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$.

Outline

Permutations, LRmins

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Inversions, Coinversions

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$\sigma = 24531$
$\mathrm{inv}(\sigma) = 6 \quad \{(2,1),(4,3),(4,1),(5,3),(5,1),(3,1)\}$
$\mathrm{coinv}(\sigma) = 4 \quad \{(2,4),(2,5),(2,3),(4,5)\}$

Reduction of A Sequence

Given a sequence of distinct positive integers $w = w_1 \dots w_n$, we let the reduction (or standardization) of the sequence, $\mathrm{red}(w)$, denote the permutation of $[n]$ obtained from w by replacing the i-th smallest letter in w by i.

Example

If $w = 4592$, then $\mathrm{red}(w) = 2341$.

Classical Patterns Occurrence and Avoidance

- Given a permutation $\tau = \tau_1 \dots \tau_j$ in S_j,
- we say the pattern τ occurs in $\sigma = \sigma_1 \dots \sigma_n \in \mathcal{S}_n$ if there exist $1 \leq i_1 < \cdots < i_j \leq n$ such that $\mathrm{red}(\sigma_{i_1} \dots \sigma_{i_j}) = \tau$.

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$\pi = 867932451$ avoids pattern 132, contains pattern 123.
$\mathrm{occr}_{123}(\pi) = 2$ since pattern occurrences are $6, 7, 9$ and $3, 4, 5$.

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- inversion $\longrightarrow$ pattern 21, coinversion $\longrightarrow$ pattern 12.

$\mathcal{S}_n(\sigma)$

- We let $\mathcal{S}_n(\lambda)$ denote the set of permutations in $\mathcal{S}_n$ avoiding λ.
- Let $\Lambda = \{\lambda_1, \dots, \lambda_r\}$, then $\mathcal{S}_n(\Lambda)$ is the set of permutations in $\mathcal{S}_n$ avoiding $\lambda_1, \dots, \lambda_r$.

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- $|\mathcal{S}_n(132)| = |\mathcal{S}_n(123)| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the n^{th} Catalan number.
- C_n is also the number of $n \times n$ Dyck paths.

Our Problem

Given two sets of permutations $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ and $\Gamma = \{\gamma_1, \ldots, \gamma_s\}$, we study the distribution of classical patterns $\gamma_1, \ldots, \gamma_s$ in $\mathcal{S}_n(\Lambda)$.

Especially, we study pattern τ distribution in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$ in the case when τ is of length 3 and some special form.

Generating Function

For $\Gamma = \{\gamma_1, \ldots, \gamma_s\}$, we define

Generating functions $Q^{\Gamma}_{n,\Lambda}$, Q^{Γ}_{Λ}

$$Q^{\Gamma}_{n,\Lambda}(x_1, \ldots, x_s) = \sum_{\sigma \in \mathcal{S}_n(\Lambda)} x_1^{\mathrm{occr}_{\gamma_1}(\sigma)} \cdots x_s^{\mathrm{occr}_{\gamma_s}(\sigma)}, \quad \text{and}$$

$$\begin{aligned} Q^{\Gamma}_{\Lambda}(t, x_1, \ldots, x_s) &= 1 + \sum_{n \geq 1} t^n Q^{\Gamma}_{n,\Lambda}(x_1, \ldots, x_s) \\ &= 1 + \sum_{n \geq 1} t^n \sum_{\sigma \in \mathcal{S}_n(\Lambda)} x_1^{\mathrm{occr}_{\gamma_1}(\sigma)} \cdots x_s^{\mathrm{occr}_{\gamma_s}(\sigma)}. \end{aligned}$$

Generating Function

Especially, we have

Generating functions $Q_{n,\lambda}^{\gamma}$, Q_{λ}^{γ}

$$Q_{n,\lambda}^{\gamma}(x) = \sum_{\sigma \in \mathcal{S}_n(\lambda)} x^{\mathrm{occr}_{\gamma}(\sigma)} \quad \text{and}$$

$$Q_{\lambda}^{\gamma}(t,x) = 1 + \sum_{n \geq 1} t^n Q_{n,\lambda}^{\gamma}(x) = 1 + \sum_{n \geq 1} t^n \sum_{\sigma \in \mathcal{S}_n(\lambda)} x^{\mathrm{occr}_{\gamma}(\sigma)}.$$

Wilf-equivalence

Given a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in \mathcal{S}_n$,

- reverse : $\sigma^r = \sigma_n \ldots \sigma_2\sigma_1$,
- complement : $\sigma^c = (n+1-\sigma_1)(n+1-\sigma_2)\ldots(n+1-\sigma_n)$,
- reverse-complement : $\sigma^{rc} = (\sigma^r)^c$,
- inverse : σ^{-1}.

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Example

Let $\sigma = 15324$, then
$\sigma^r = 42351,\ \sigma^c = 51342,\ \sigma^{rc} = 24315,\ \sigma^{-1} = 14352.$

Wilf-equivalence

Lemma

Given any permutation pattern γ*,*

$$Q_{\lambda}^{\gamma}(t,x) = Q_{\lambda^*}^{\gamma^*}(t,x),$$

where $*$ *is* r, c, rc *or* -1.

reason: e.g. $\sigma \in \mathcal{S}_n(\lambda) \longleftrightarrow \sigma^r \in \mathcal{S}_n(\lambda^r)$,

$$\mathrm{occr}_{\gamma}(\sigma) = \mathrm{occr}_{\gamma^r}(\sigma^r).$$

Wilf-equivalence

Since $123 = 123^{rc} = 123^{-1}$ and $132 = 132^{-1}$, we have the following corollary.

Corollary

Given any permutation pattern γ,

$$Q_{123}^{\gamma}(t,x) = Q_{123}^{\gamma^{rc}}(t,x) = Q_{123}^{\gamma^{-1}}(t,x),$$

$$Q_{132}^{\gamma}(t,x) = Q_{132}^{\gamma^{-1}}(t,x).$$

Wilf-equivalence

When we let γ be a pattern of length 3,

Corollary

There are 4 *Wilf-equivalent classes for* $\mathcal{S}_n(132)$,

(1) $Q_{132}^{123}(t,x)$,
(2) $Q_{132}^{213}(t,x)$,
(3) $Q_{132}^{231}(t,x) = Q_{132}^{312}(t,x)$,
(4) $Q_{132}^{321}(t,x)$,

and there are 3 *Wilf-equivalent classes for* $\mathcal{S}_n(123)$,

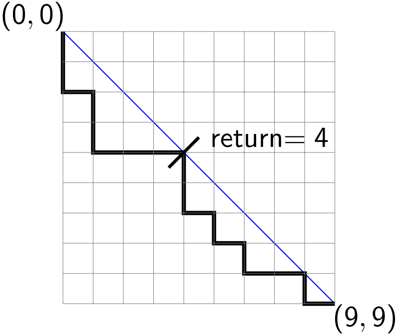
(1) $Q_{123}^{132}(t,x) = Q_{123}^{213}(t,x)$,
(2) $Q_{123}^{231}(t,x) = Q_{123}^{312}(t,x)$,
(3) $Q_{123}^{321}(t,x)$.

Method – Using Dyck Path Bijections

An (n, n)-Dyck path is a path from $(0, 0)$ to (n, n) that stays on or below the diagonal $y = x$.

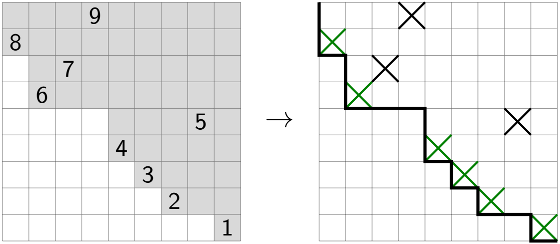
The return of a Dyck path P is the smallest number $i > 0$ such that P goes through the point (i, i).

Example: a $(9, 9)$-Dyck path.



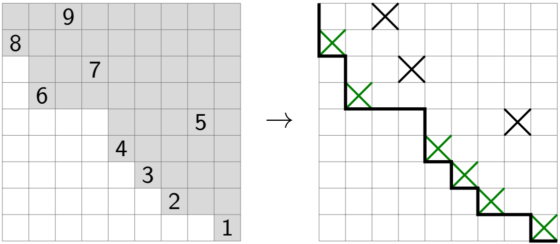
Method – Using Dyck Path Bijections

Krattenthaler's map $\Phi : S_n(132) \to D_n$.



Method – Using Dyck Path Bijections

Elizalde and Deutsch's map $\Psi : \mathcal{S}_n(123) \to \mathcal{D}_n$.

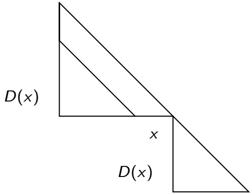


Method – Using Dyck Path Bijections

Then, we the recursion of Dyck path by breaking the path at the first place it hits the diagonal to break it into 2 Dyck paths.

Let $D(x)$ be the generating function enumerating the number of Dyck paths of size n,

$$D(x) = 1 + xD(x)^2.$$

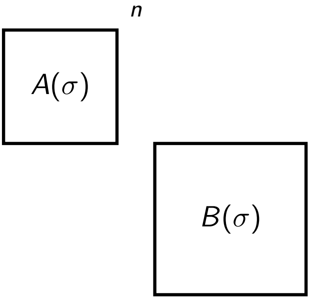


Recursion of Dyck path

Method – Recursive Counting for $\mathcal{S}_n(132)$

Let $\sigma = \sigma_1 \cdots \sigma_n \in \mathcal{S}_n(132)$ such that $\sigma_k = n$. The numbers $\sigma_1, \ldots, \sigma_{k-1}$ must be bigger than the numbers $\sigma_{k+1}, \ldots, \sigma_n$.

We let $A(\sigma) = \mathrm{red}(\sigma_1 \cdots \sigma_{k-1})$ and $B(\sigma) = \mathrm{red}(\sigma_{k+1} \cdots \sigma_n)$, then $A(\sigma) \in \mathcal{S}_{k-1}(132)$ and $B(\sigma) \in \mathcal{S}_{n-k}(132)$.



Outline

Counting Length 2 pattern in $\mathcal{S}_n(132)$

We first consider permutations that are avoiding 132 and the distribution of pattern of length 2, i.e. inv and coinv.

Let $Q_n(x_1, x_2) := Q_{n,132}^{\{12,21\}}(x_1, x_2)$,
$Q(t, x_1, x_2) := Q_{132}^{\{12,21\}}(t, x_1, x_2)$.

Counting Length 2 pattern in $\mathcal{S}_n(132)$

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$Q(t,x_1,x_2) := Q_{132}^{\{12,21\}}(t,x_1,x_2)$.

Theorem (Fürlinger and Hofbauer)

$$
\begin{aligned}
Q_0(x_1,x_2) &= 1 \\
Q_n(x_1,x_2) &= \sum_{k=1}^{n} x_1^{k-1} x_2^{k(n-k)} Q_{k-1}(x_1,x_2) Q_{n-k}(x_1,x_2),
\end{aligned}
$$

and

$$Q(t,x,1) = 1 + tQ(t,x,1) \cdot Q(tx,x,1).$$

$Q_n(q,1)$ is q-Catalan number.

Counting Length 3 pattern in $\mathcal{S}_n(132)$

Let $\Gamma_2 = \{12, 21\}$ and $\Gamma_3 = \{123, 213, 231, 312, 321\}$ be sets of permutation patterns. We shall prove the following theorem about the function

$$Q_{n,132}^{\Gamma_2 \cup \Gamma_3}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = Q_{n,132}^{\{12,21,123,213,231,312,321\}}(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$$

Counting Length 3 pattern in $\mathcal{S}_n(132)$

Theorem

The function $Q_{n,132}^{\Gamma_2 \cup \Gamma_3}(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ *satisfies the recursion*

$$Q_0(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = 1,$$

$$Q_n(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = \sum_{k=1}^{n} x_1^{k-1} x_2^{k(n-k)} x_5^{(k-1)(n-k)} \cdot Q_{k-1}(x_1 x_3 x_5^{(n-k)}, x_2 x_4 x_7^{(n-k)}, x_3, x_4, x_5, x_6, x_7) \cdot Q_{n-k}(x_1 x_6^{k}, x_2 x_7^{k}, x_3, x_4, x_5, x_6, x_7).$$

Counting Length 3 pattern in $\mathcal{S}_n(132)$

Let

$$P_n^{\gamma}(q,x) := \sum_{\sigma\in\mathcal{S}_n(132)} q^{\mathrm{coinv}(\sigma)} x^{\mathrm{occr}_{\gamma}(\sigma)},$$

then

$$P_0^{\gamma}(q,x) = 1 \quad \text{for each pattern } \gamma, \textit{and}$$

Counting Length 3 pattern in $\mathcal{S}_n(132)$

Corollary

We have the following equations.

$$
\begin{aligned}
P_n^{123}(q,x) &= \sum_{k=1}^{n} q^{k-1} P_{k-1}(qx,x) P_{n-k}(q,x),\\
P_n^{213}(q,x) &= \sum_{k=1}^{n} q^{k-1} x^{\frac{(k-1)(k-2)}{2}} P_{k-1}(\frac{q}{x},x) P_{n-k}(q,x),\\
P_n^{231}(q,x) &= \sum_{k=1}^{n} q^{k-1} x^{(k-1)(n-k)} P_{k-1}(qx^{(n-k)},x) P_{n-k}(q,x),\\
P_n^{321}(q,x) &= \sum_{k=1}^{n} q^{k-1} x^{\frac{(n-k)(kn-4k+2)}{2}} P_{k-1}(\frac{q}{x^{n-k}},x) P_{n-k}(\frac{q}{x^k},x).
\end{aligned}
$$

Track all patterns of length 2 and 3 in $\mathcal{S}_n(132)$

Expansion of $Q_{n,132}^{\{12,21,123,213,231,312,321\}}(x_1,x_2,x_3,x_4,x_5,x_6,x_7)$

n	$Q_{n,132}^{12,21,123,213,231,312,321}(x_1,x_2,x_3,x_4,x_5,x_6,x_7)$
0	1
1	1
2	x_1+x_2
3	$x_1^3x_7+x_1^2x_2x_5+x_1^2x_2x_6+x_1x_2^2x_4+x_2^3x_3$
4	$x_1^6x_7^4+x_1^5x_2x_5^2x_7^2+x_1^5x_2x_5x_6x_7^2+x_1^5x_2x_6^2x_7^2+x_1^4x_2^2x_4x_5^2x_7+x_1^4x_2^2x_4x_6^2x_7+x_1^4x_2^2x_5^2x_6^2$ $+x_1^3x_2^3x_3x_5^3+x_1^3x_2^3x_3x_6^3+x_1^3x_2^3x_4^3x_7+x_1^2x_2^4x_3x_4^2x_5+x_1^2x_2^4x_3x_4^2x_6+x_1x_2^5x_3^2x_4^2+x_2^6x_3^4$
5	$x_1^{10}x_7^{10}+x_1^9x_2x_5^3x_7^7+x_1^9x_2x_5^2x_6x_7^7+x_1^9x_2x_5x_6^2x_7^7+x_1^9x_2x_6^3x_7^7+x_1^8x_2^2x_4x_5^4x_7^5+x_1^8x_2^2x_4x_5^2x_6^2x_7^5$ $+x_1^8x_2^2x_4x_6^4x_7^5+x_1^8x_2^2x_5^4x_6^2x_7^4+x_1^8x_2^2x_5^3x_6^3x_7^4+x_1^8x_2^2x_5^2x_6^4x_7^4+x_1^7x_2^3x_3x_5^6x_7^3+x_1^7x_2^3x_3x_5^3x_6^3x_7^3$ $+x_1^7x_2^3x_3x_6^6x_7^3+x_1^7x_2^3x_4^3x_5^3x_7^4+x_1^7x_2^3x_4^3x_6^3x_7^4+x_1^7x_2^3x_4x_5^4x_6^3x_7^2+x_1^7x_2^3x_4x_5^3x_6^4x_7^2+x_1^6x_2^4x_3x_4^2x_5^5x_7^2$ $+x_1^6x_2^4x_3x_4^2x_5^4x_6x_7^2+x_1^6x_2^4x_3x_4^2x_5x_6^4x_7^2+x_1^6x_2^4x_3x_4^2x_6^5x_7^2+x_1^6x_2^4x_3x_5^6x_6^3+x_1^6x_2^4x_3x_5^3x_6^6$ $+x_1^6x_2^4x_4^6x_7^4+x_1^5x_2^5x_3^2x_4^2x_5^5x_7+x_1^5x_2^5x_3^2x_4^2x_6^5x_7+x_1^5x_2^5x_3x_4^5x_5^2x_7^2+x_1^5x_2^5x_3x_4^5x_5x_6x_7^2$ $+x_1^5x_2^5x_3x_4^5x_6^2x_7^2+x_1^4x_2^6x_3^4x_5^6+x_1^4x_2^6x_3^4x_6^6+x_1^4x_2^6x_3^2x_4^5x_5^2x_7+x_1^4x_2^6x_3^2x_4^5x_6^2x_7+x_1^4x_2^6x_3^2x_4^4x_5^2x_6^2$ $+x_1^3x_2^7x_3^4x_4^3x_5^3+x_1^3x_2^7x_3^4x_4^3x_6^3+x_1^3x_2^7x_3^3x_4^6x_7+x_1^2x_2^8x_3^5x_4^4x_5+x_1^2x_2^8x_3^5x_4^4x_6+x_1x_2^9x_3^7x_4^3+x_2^{10}x_3^{10}$

Track all patterns of length 2, 3 and 4 in $\mathcal{S}_n(132)$

Let $\Gamma_4 = \mathcal{S}_4(132)$. We want to compute $Q_{n,132}^{\Gamma_2 \cup \Gamma_3 \cup \Gamma_4}(x_1, \ldots, x_{21})$. We shall do a refinement:

Track all patterns of length 2, 3 and 4 in $\mathcal{S}_n(132)$

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$$Q_{n,i}(x_1,\ldots,x_{19}) := Q_{n,132}^{\Gamma_2\cup\Gamma_3\cup\Gamma_4}(x,1,x_1,\ldots,x_{19})\Big|_{x^i},$$

then

$$Q_{n,132}^{\Gamma_2\cup\Gamma_3\cup\Gamma_4}(x_1,\ldots,x_{21}) = \sum_{i=0}^{\binom{n}{2}} x_1^i x_2^{\binom{n}{2}-i} Q_{n,i}(x_3,\ldots,x_{21}).$$

Track all patterns of length 2, 3 and 4 in $\mathcal{S}_n(132)$

Theorem

$$Q_{0,0}(x_1,\ldots,x_5,y_1,\ldots,y_{14}) = 1,$$

$$Q_{n,i}(x_1,\ldots,x_5,y_1,\ldots,y_{14}) = 0 \text{ for } i < 0 \text{ or } i > \binom{n}{2}, \text{ and}$$

Track all patterns of length 2, 3 and 4 in $\mathcal{S}_n(132)$

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$$Q_{n,i}(x_1,\ldots,x_5,y_1,\ldots,y_{14}) = \sum_{k=1}^{n}\sum_{j=0}^{i+1-k} x_1^j x_2^{\binom{k-1}{2}-j}$$

$$x_3^{(n-k)(k+j-1)} x_4^{k(i+1-k-j)} x_5^{(n-k)\left(\binom{k-1}{2}-j\right)+k\left(\binom{n-k}{2}+k+j-i-1\right)}$$

$$y_4^{j(n-k)} y_7^{\left(\binom{k-1}{2}-j\right)(n-k)} y_8^{(j+k-1)(i+1-k-j)} y_9^{(j+k-1)\left(\binom{n-k}{2}+k+j-i-1\right)}$$

$$y_{13}^{\left(\binom{k-1}{2}-j\right)(i+1-k-j)} y_{14}^{\left(\binom{k-1}{2}-j\right)\left(\binom{n-k}{2}+k+j-i-1\right)} \cdot Q_{k-1,j}(x_1y_1y_4^{n-k},$$

$$x_2y_2y_7^{n-k}, x_3y_3y_9^{n-k}, x_4y_5y_{12}^{n-k}, x_5y_6y_{14}^{n-k}, y_1,\ldots,y_{14})$$

$$\cdot\, Q_{n-k,i+1-k-j}(x_1y_{10}^k, x_2y_{11}^k, x_3y_{12}^k, x_4y_{13}^k, x_5y_{14}^k, y_1,\ldots,y_{14}).$$

Special Case: $\gamma = 1 \cdots m$ for $\mathcal{S}_n(132)$

Given $m \geq 2$ and $n \geq 0$, we let

$$Q^{(m)}_{n,132}(x_2, x_3, \ldots, x_m) := \sum_{\sigma \in \mathcal{S}_n(132)} x_2^{\mathrm{occr}_{12}(\sigma)} x_3^{\mathrm{occr}_{123}(\sigma)} \cdots x_m^{\mathrm{occr}_{12\cdots m}(\sigma)}$$

$$Q^{(m)}_{132}(t, x_2, x_3, \ldots, x_m) := \sum_{n \geq 0} t^n Q^{(m)}_{n,132}(x_2, x_3, \ldots, x_m).$$

Special Case: $\gamma = 1 \cdots m$ for $\mathcal{S}_n(132)$

Theorem

$$Q_{n,132}^{(m)}(x_2, \dots, x_m) = \sum_{k=1}^{n} x_2^{k-1}$$
$$\cdot\, Q_{k-1,132}^{(m)}(x_2x_3, x_3x_4, \dots, x_{m-1}x_m, x_m)\, Q_{n-k,132}^{(m)}(x_2, \dots, x_m),$$

$$Q_{132}^{(m)}(t, x_2, \dots, x_m) = 1 + t$$
$$\cdot\, Q_{132}^{(m)}(tx_2, x_2x_3, x_3x_4, \dots, x_{m-1}x_m, x_m) \cdot Q_{132}^{(m)}(t, x_2, \dots, x_m).$$

Outline

Counting Length 3 pattern in $\mathcal{S}_n(123)$

We also get nice recursions for patterns 132 and 231 distributions in $\mathcal{S}_n(123)$.

Theorem (when $\gamma = 132$)

Let $Q_{n,123}^{132}(s,q,x) = \sum_{\sigma \in \mathcal{S}_n(123)} s^{LRmin(\sigma)} q^{\mathrm{coinv}(\sigma)} x^{\mathrm{occr}_{132}(\sigma)}$, *then we have the following recursions,*

$$Q_{0,123}^{132}(s,q,x) = 1,$$

$$Q_{n,123}^{132}(s,q,x) = sQ_{n-1} + \sum_{k=2}^{n} Q_{k-1}(sq,qx,x)Q_{n-k}(s,q,x).$$

Counting Length 3 pattern in $\mathcal{S}_n(123)$

Given $\sigma \in \mathcal{S}_n(123)$, we let $\mathrm{linv}(\sigma)$ be the number of pairs (i,j) such that σ_i is a left-to-right minimum, σ_j is not a left-to-right minimum and $\sigma_i > \sigma_j$.

Counting Length 3 pattern in $\mathcal{S}_n(123)$

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$$D_n(s, q, x, y) := \sum_{\sigma \in \mathcal{S}_n(123)} s^{\mathrm{LRmin}(\sigma)} q^{\mathrm{occr}_{12}(\sigma)} x^{\mathrm{linv}(\sigma)} y^{\mathrm{occr}_{231}(\sigma)},$$

$$D_{n,k}(q, x, y) := \sum_{\sigma \in \mathcal{S}_n(123), \mathrm{LRmin}(\sigma) = k} q^{\mathrm{occr}_{12}(\sigma)} x^{\mathrm{linv}(\sigma)} y^{\mathrm{occr}_{231}(\sigma)}.$$

Counting Length 3 pattern in $\mathcal{S}_n(123)$

Theorem

$D_0(s,q,x,y) = D_{0,0}(q,x,y) = 1$. *For any* $n,k \geq 1$,
$D_{n,1}(q,x,y) = q^{n-1}$, $D_{n,n}(q,x,y) = 1$,
$D_{n,k}(q,x,y) = 0$ *for* $k > n$, *and*

$$D_{n,k}(q,x,y) = x^{n-k} D_{n-1,k-1}(q,x,y) + q^k D_{n-1,k}(q,xy,y) + \sum_{i=2}^{n-1} \sum_{j=\max(1,k+i-n)}^{\min(i-1,k-1)} q^j x^{j(n-i-k+j)} y^{j(n-i)} \cdot D_{i-1,j}(qy^{n-i},xy,y) D_{n-i,k-j}(q,x,y).$$

Special Case: $\gamma = 1 \cdots m$ for $\mathcal{S}_n(123)$

We define

$$Q_{n,123}^{(m)}(s, x_2, \ldots, x_m) := \sum_{\sigma \in \mathcal{S}_n(123)} s^{\mathrm{LRmin}(\sigma)} x_2^{\mathrm{occr}_{12}(\sigma)} \cdots x_m^{\mathrm{occr}_{1m(m-1)\cdots 2}(\sigma)},$$

$$Q_{123}^{(m)}(t, s, x_2, \ldots, x_m) := \sum_{n \geq 0} t^n Q_{n,123}(s, x_2, x_3, \ldots, x_m).$$

Special Case: $\gamma = 1 \cdots m$ for $\mathcal{S}_n(132)$

Theorem

$$Q^{(m)}_{n,123}(s, x_2, \ldots, x_m) = sQ^{(m)}_{n-1,123}(s, x_2, \ldots, x_m) + \sum_{k=2}^{n} Q^{(m)}_{k-1,123}(sx_2, x_2x_3, x_3x_4, \ldots, x_{m-1}x_m, x_m) \cdot Q^{(m)}_{n-k,123}(s, x_2, \ldots, x_m),$$

and

$$Q^{(m)}_{123}(t, s, x_2, \ldots, x_m) = 1 + t(s-1)Q^{(m)}_{123}(t, s, x_2, \ldots, x_m) + tQ^{(m)}_{123}(t, sx_2, x_2x_3, x_3x_4, \ldots, x_{m-1}x_m, x_m) \cdot Q^{(m)}_{123}(t, s, x_2, \ldots, x_m).$$

An equality between $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$

By looking at the coefficients of the generating functions, we find a coincidence among $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$. We have the following theorem.

Theorem

For any nonnegative integers $i < j$,

$$[t^n x^i]_{Q_{132}^{1\cdots j}(t,x)} = [t^n x^i]_{Q_{123}^{1j\cdots 2}(t,x)}.$$

Outline

Applications in Pattern Popularity

Let S be a set of permutations and γ be a permutation pattern. The *popularity* of γ in S, $f_S(\gamma)$, is defined by

$$f_S(\gamma) := \sum_{\sigma \in S} \mathrm{occr}(\gamma).$$

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Let

$$F_\gamma(t) := \sum_{n\geq 0} f_{\mathcal{S}_n(132)}(\gamma)t^n \quad \text{and}$$

$$G_\gamma(t) := \sum_{n\geq 0} f_{\mathcal{S}_n(123)}(\gamma)t^n,$$

Applications in Pattern Popularity

Bóna and Homberger studied the popularity of length 2 or 3 patterns in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$.

Theorem (Bóna and Homberger)

Let $C(t) := \sum_{n\geq 0} C_n t^n$ be the generating function of Catalan numbers. Then

$$\begin{aligned} F_{12}(t) &= \frac{t^2C^3(t)}{(1-2tC(t))^2}, \\ G_{12}(t) &= \frac{tC^2(t)}{1-2tC(t)}. \end{aligned}$$

Applications in Pattern Popularity

Our results implies the following.

Theorem

Let $m > 2$ be an integer. Then

$$F_{12\cdots m}(t) = \frac{tC(t)F_{12\cdots(m-1)}(t)}{1-2tC(t)}, \quad \textit{and}$$

$$G_{1m\cdots 2}(t) = \frac{tC(t)G_{1(m-1)\cdots 2}(t)}{1-2tC(t)}.$$

Outline

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- Circular permutations: permutations with one cycle. $\mathcal{CS}_n$: the set of size n circular permutations.

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- $\mathrm{coccr}_\gamma(\sigma)$: total occurrence of γ in all expressions $\sigma_i \cdots \sigma_n \sigma_1 \cdots \sigma_{i-1}$ for any $i = 1, \ldots, n$.
- $\mathcal{CS}_n(\lambda)$ when $|\lambda| = 1, 2$ or 3 are trivial.
- By symmetry, we only need to study circular pattern distribution in $\mathcal{CS}_n(1234)$, $\mathcal{CS}_n(1243)$ and $\mathcal{CS}_n(1324)$ when $|\lambda| = 4$.

Circular Pattern Distribution in $\mathcal{CS}_n(1243)$

Let $P_{n,1243}(y_{123}, y_{132}, y_{1234}, y_{1324}, y_{1342}, y_{1423}, y_{1432}) :=$

$$\sum_{\sigma\in\mathcal{CS}_n(1243)} y_{123}^{\mathrm{coccr}_{123}(\sigma)} y_{132}^{\mathrm{coccr}_{132}(\sigma)} y_{1234}^{\mathrm{coccr}_{1234}(\sigma)} y_{1324}^{\mathrm{coccr}_{1324}(\sigma)} \cdot y_{1342}^{\mathrm{coccr}_{1342}(\sigma)} y_{1423}^{\mathrm{coccr}_{1423}(\sigma)} y_{1432}^{\mathrm{coccr}_{1432}(\sigma)}.$$

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Theorem

For any $n \geq 1$,

$$\begin{aligned} &P_{n,1243}(y_{132}, y_{1234}, y_{1324}, y_{1342}, y_{1423}, y_{1432}) \\ &= Q_{n-1,132}^{\Gamma_2 \cup \Gamma_3 \cup \Gamma_4}(y_{123}, y_{132}, y_{123}\ y_{1234}, y_{132}\ y_{1324}, y_{123}\ y_{1342}, \\ &\quad y_{123}\ y_{1423}, y_{132}\ y_{1432}, y_{1234}, y_{1342}, y_{1423}, y_{1234}, 0, y_{1432}, \\ &\quad y_{1324}, y_{1234}, y_{1342}, y_{1234}, y_{1342}, y_{1423}, 0, y_{1432}). \end{aligned}$$

Circular Pattern Distribution in $\mathcal{CS}_n(1324)$

Let $P_{n,1324}(y_{123}, y_{132}, y_{1234}, y_{1243}, y_{1342}, y_{1423}, y_{1432}) :=$

$$\sum_{\sigma \in \mathcal{CS}_n(1324)} y_{123}^{\mathrm{coccr}_{123}(\sigma)} y_{132}^{\mathrm{coccr}_{132}(\sigma)} y_{1234}^{\mathrm{coccr}_{1234}(\sigma)} y_{1243}^{\mathrm{coccr}_{1243}(\sigma)} \cdot y_{1342}^{\mathrm{coccr}_{1342}(\sigma)} y_{1423}^{\mathrm{coccr}_{1423}(\sigma)} y_{1432}^{\mathrm{coccr}_{1432}(\sigma)},$$

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Theorem

For any $n \geq 1$,

$$P_{n,1324}(y_{132}, y_{1234}, y_{1243}, y_{1342}, y_{1423}, y_{1432}) = Q_{n-1,132}^{\Gamma_2 \cup \Gamma_3 \cup \Gamma_4}(y_{123}, y_{132}, y_{123}\ y_{1234}, y_{132}\ y_{1342}, y_{123}\ y_{1423}, y_{123}\ y_{1243}, y_{132}\ y_{1432}, y_{1234}, y_{1342}, y_{1423}, y_{1234}, y_{1243}, y_{1432}, 0, y_{1234}, y_{1342}, y_{1234}, y_{1342}, y_{1423}, y_{1243}, y_{1432}).$$

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- Pattern 321 distribution in $\mathcal{S}_n(123)$? Longer patterns in $\mathcal{S}_n(123)$?

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- $\mathcal{S}_n(\lambda)$ when $|\lambda| \geq 4$?

Thank You!